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CASE FILE COPY

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Prepared for the

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ABSTRACT

The dynamic response of nonlinear elastic structures is obtained by a step-by-step, numerical integration procedure. Two general types of systems are considered — multi-degree-of-freedom discrete systems and continuous systems of infinite degrees of freedom. The response of a shallow arch is determined to illustrate the application of the procedure to continuous systems. Linear viscous damping is included and no restrictions are placed on the variation of the excitations in space or time. Free or undamped motion is a special case of the general formulation. The effect of any properly posed set of initial conditions can be evaluated since the numerical results represent the total response including transients. The feasibility of treating large numbers of modes or degrees of freedom is limited only by computer storage and processing time.

I. INTRODUCTION

The theory of nonlinear oscillations has its origins in the nineteenth century problems of celestial mechanics. Since that time, the theory has found application in many diverse disciplines. The importance of the theory lies in the fact that most physical systems are, by nature, nonlinear when some of the simplifying restrictions are eased. Due to the mathematical complexities involved, exact closed-form solutions are known only for the simplest nonlinear equations.

Since the theory began to develop, approximate methods have been devised for solving the various problems which have arisen. These approximate methods, while yielding valuable information about the behavior of certain nonlinear oscillatory systems, are not well suited for all types of problems. They do not, in general, yield quantitative results for transient response. Furthermore, the convergence of many of the approximate methods yielding steady-state response, requires restrictions on the magnitudes of certain coefficients, among them damping. When damping is assumed negligible or small, the transients are either non-vanishing or slowly vanishing and thus, often of considerable importance. Indeed, the initial conditions, of which no account is taken by many approximate methods, may determine the type of steady-state response for a given system. And finally, the approximate methods become extremely cumbersome, if applicable at all, in the case of multi-degree-of-freedom systems with several different nonlinearities.

II. PURPOSE AND SCOPE

It is sought to develop a unified presentation of the numerical solution of any nonlinear, elastic, vibrating structure. The structure may be framed or idealized as such by lumped masses or finite elements. Continuous structures are also considered by treating the governing partial differential equation directly. As an example of such a system, the response of a shallow arch to impulse loading is presented. The number of degrees of freedom considered for any particular problem is limited only by computer storage and processing time.

A limited number of numerical examples are presented to demonstrate the simplicity with which complex problems can be solved by this approach. It is attempted, where possible, to compare the numerical results with solutions obtained by other methods of analysis. In some cases, only qualitative information is available. It is also sought to show the solutions to systems difficult or impossible to solve by other methods.

The determination of regions of instability shall be considered beyond the scope of this treatment. It will be shown however, that instability is detected when present. Self-oscillatory systems are likewise excluded from the presentation although not necessarily from the method.

III. HISTORICAL REVIEW

1. Background

Early impetus to the study of nonlinear oscillations was furnished by eighteenth century astronomers who discovered, with the aid of improved telescopes, that planetary motions differed slightly from predicted elliptical orbits. The determination of these perturbations, which are due to the gravitational attraction of other bodies in the solar system, became one of the principal problems of celestial mechanics. Subsequently, nonlinear oscillatory phenomina have manifested themselves in a wide variety of fields, notably electronics and circuit theory as well as mechanical and structural vibrations.

The mathematical analysis of the associated nonlinear differential equations has progressed in two distinct directions, namely the topological or graphical methods which yield qualitative results, and the analytical methods which yield quantitative results. All the methods within these two categories are approximate except for a few exact solutions which have been found for the simplest nonlinear equations. Furthermore, any one method of solution gives results of a limited, specialized nature and therefore it is often necessary to use two or more methods in conjunction with each other to gain deeper insight into the behavior of a given system.

Both the topological and the analytical approaches evolved from the work of Poincaré in the late nineteenth century. Poincaré established the existence of periodic solutions, which he referred to as limit cycles, for a class of nonlinear differential equations. He also introduced the method of small parameters, commonly called the perturbation method, for treating slightly nonlinear equations.

Certain periodic solutions of one such ''almost linear'' equation were studied by Duffing [21] in 1918. Duffing obtained harmonic solutions, that is those whose frequency is equal to that of the excitation, by an iteration method using the linear, undamped, free vibration as a first approximation to the solution.

The advent of electronics furthered interest in nonlinear oscillations. In 1920, van der Pol formulated and analyzed the nonlinear differential equation of the electron tube oscillator. He concluded from his graphical analysis that the equation possessed a periodic solution.

Nine years later, Andronow correlated van der Pol's results with the work of Poincaré and in the decade that followed, many important contributions to the field were made by Russian researchers. Probably the most prominent among these contributions was the asymptotic method of Krylov and Bogoliubov [17]. This is based on the assumption of a solution having the form of simple harmonic motion but with slowly varying amplitude and phase. This same idea in a slightly less convenient form is also due to van der Pol.

The methods just mentioned, the perturbation method, Duffing's method and the asymptotic method represent three classical, analytical methods available for treating nonlinear differential equations with small nonlinearities.

2. Classical Methods

a) Analytical - Most of the analytical methods which have been traditionally used to treat nonlinear equations, may be classified as being similar to one of the three main methods just mentioned. The advantage of using one of these analytical methods lies in the fact that the solution is obtained in the form of an algebraic expression, valid in some region of the parameter space. The effect of varying parameters throughout this region can thus be readily determined from the solution itself. On the other hand, graphical and numerical procedures, which will be discussed later, require that all the parameters be given numerical values and any variation of these values necessitates another complete solution. Any parameter studies by these methods are therefore extremely tedious.

Many of the iterative procedures which have been devised, similar to Duffing's method, are based on what is sometimes called the principle of "harmonic balance". According to this principle, an approximate solution of first-order accuracy is obtained if only the component of fundamental frequency is retained and the constants involved are adjusted to satisfy all terms of fundamental frequency in the equation. Frequently, in a Fourier

Series representation of a periodic oscillation, only the fundamental component and possibly one or two harmonics are of sufficient amplitude to make a significant contribution to the total response. Coefficients of higher-order components can similarly be adjusted to satisfy all terms of their respective frequencies and the inclusion of these components yields correspondingly higher-order approximations.

Using this principle, Duffing [21] studied the harmonic solutions to the nonlinear differential equation which is now commonly called Duffing's equation. The equation has a cubic nonlinearity and a harmonic forcing function both of which are multiplied by a small parameter. As this parameter approaches zero, the equation becomes linear and homogeneous. Duffing therefore, used the solution of the linear, homogeneous equation to initialize his iteration procedure. Substitution of this approximate solution and its devivatives into the equation and equating coefficients of terms of fundamental frequency according to the principle of harmonic balance, yields a first-order relation between frequency of excitation and amplitude of response. Solutions and amplitude-frequency relations of higher-order accuracy can be obtained by assuming a solution of higher harmonics and similarly equating coefficients of higher-order terms. Duffing obtained the same results by substituting the first approximate solution for only the lower order derivatives of the differential equation and then integrating twice to obtain the next higher approximation. The amplitude of the fundamental frequency term is then equated to the fundamental amplitude of the first approximation. This process can then be continued to obtain successively better approximations and at each step of the iteration procedure, the amplitude of the leading term is taken as fixed by the lowest approximation. While such iteration procedures are relatively simple to apply, their mathematical foundation is not rigorous and consequently, questions of convergence, error estimates and ranges of applicability are difficult to resolve. Another disadvantage involved in the use of iteration methods is that some prior knowledge concerning the type of solution to expect is necessary in order to choose a good first approximation. This difficulty can sometimes be overcome by obtaining the first approximation from a preliminary analysis by some other method.

Another method of successive approximations which utilizes the principle of harmonic balance is the asymptotic method of Krylov and Bogoliubov [17], also called the method of van der Pol or the method of variation of parameters. The first approximation is taken in the form of simple harmonic motion but the amplitude and phase, which are constants of integration in linear theory, are now assumed to be slowly varying functions of time. Consequently, the first and second time derivatives of the amplitude and phase are of successively smaller orders of magnitude. This renders certain terms at various stages of the analysis negligible and utilizing the principles of harmonic balance and Fourier analysis, the first derivatives of the amplitude and phase can be obtained. This method is generally more difficult to apply than others, particularly in obtaining higher order approximations. It has however, been extended by Mitropolsky [17] to yield valuable results to a more general class of problems than can be treated by other methods. Krylov, Bogoliubov and Mitropolsky [17] also used the asymptotic methods to transform nonlinear equations with constant coefficients into linear equations with variable coefficients. While such linear equations may be just as difficult to solve, this equivalent linearization technique can be useful in certain types of problems.

The third class of analytic methods and perhaps the most common, are the perturbation methods. These methods, which vary slightly depending on the problem and the type of solution desired, are based on the construction of a solution to the slightly nonlinear problems from the known solution to the adjacent linear problem. The solution is assumed in the form of a power series in powers of the small parameter. Substitution of this perturbation series into the nonlinear equation and collecting terms of like power in the small parameter, results in a set of simultaneous, linear differential equations in the coefficients of the perturbation series, which can be solved in sequence. This method can involve computational difficulties in solving for higher order terms of the series but if the parameter is sufficiently small, only the first few terms are necessary for good accuracy. In such instances when the series converges rapidly, the perturbation methods are quite easy to apply and, unlike the iteration methods, the theoretical justification is rigorous.

All these approximate, analytical methods have their individual advantages and disadvantages but they also all have certain common limitations.

One such limitation is the qualitative knowledge concerning the solution which is generally required before the analytical procedure can be applied. Such qualitative information can often be supplied by a geometrical or topological analysis of a nonlinear equation.

b) Topoligical - Since Poincaré first used the method of the phase plane, an extensive theory involving topological methods has been developed. These methods are comparatively easy to apply and are extremely powerful in that they give useful, descriptive insights into the behavior of certain nonlinear equations, specifically with regard to periodicity and stability. One drawback to these geometrical methods lies in the fact that, with very few exceptions, only autonomous equations, that is, equations in which the independent variable "time" does not appear explicitly, can be handled. Therefore, forced oscillations can be treated by topological methods only if the nonautonomous equation can somehow be transformed into an autonomous one.

In the phase plane method, the second order equation in displacement is reduced by the transformation $v = \dot{x}(t)$ into a first order equation in velocity $v\left[x(t)\right]$. The integral curves of this equation are plotted or sketched in the x, v-plane or phase plane. These integral curves are curves of constant energy and are thus also called energy curves. If a motion is periodic, the corresponding integral curve is closed. Such a closed curve is known as a limit cycle. Geometric criteria for the existence of limit cycles have been developed so that in certain cases it can be proven that no limit cycles and thus no periodic solutions can exist.

Closely related to theory of limit cycles, is another important feature of the phase plane, namely singular points. These are points where the quantity dv/dx is indeterminate and they correspond to the equilibrium states of the physical system. The analysis of these singular or critical points determines the stability or instability of the corresponding equilibrium states.

Beside providing quick and easy qualitative information preliminary to an analytical analysis, the topological approach compliments the analytical methods in another way. In treating only periodic solutions, most analytical methods give no information concerning the response during the time before the steady-state condition was reached. The transient response can be of great importance in many problems where either the motion is of interest for only a short period of time, where damping is small so that the transients remain large for a relatively long period of time or in cases where the effect of varying initial conditions is of interest.

3. Recent Work

a) Single-degree-of-freedom systems - Recent researchers have continued to treat nonlinear oscillations by analytical and topological methods. In 1938, an iteration method for steady oscillations was presented by Rauscher (1) which, unlike the Duffing and related methods beginning with the linear, homogeneous problem, starts the iteration procedure from the nonlinear, autonomous problem. Specifically, the free, undamped, nonlinear response corresponding to an arbitrary, assumed amplitude is computed as a first approximation. Then, since a periodic solution is sought, the phase between the external force and the free motion is adjusted to exactly balance the energy lost in each cycle. Using the time-displacement relation of the first approximation, the excitation is expressed as a function of displacement and combined with the restoring and damping forces resulting in a new, effective restoring force upon which a new free vibration is based. This procedure is repeated until the time-displacement relations converge and the frequency of the forced motion at the assumed amplitude is then known. The application of this method is quite tedious by comparison with Duffing's iteration method but convergence is usually rapid and one iteration may suffice in many cases. The idea of using the free, nonlinear motion as a first approximation is attractive for a number of reasons. First, the equation is exactly, although not always easily, integrable by means of elliptic functions and second, since it is closer to the forced motion than is the homogeneous problem, convergence is improved. Also, restrictions on the size of certain small parameters are eased. At the time that Rauscher's iteration method was presented, the autonomous problem had also been used to generate perturbation solutions [18, 19, 20].

Ever since the classic work of Duffing [21], the Ritz method was also being applied to a number of nonlinear vibrations problems [2, 19, 22]. In general, the methods of Ritz and Galerkin lead to a number of simultaneous, algebraic equations in a number of unknown constants. In the case of a nonlinear problem, these equations are also nonlinear and thus, a final algebraic solution to more than one or two terms is difficult, but frequently much information can be gained by examining certain special cases for which the equations simplify. The Ritz method as usually constituted, requires the formulation of a variational principle while the Galerkin method requires knowledge of the differential equation and, as with many other analytical methods, some prior knowledge concerning the behavior of the system is helpful in arriving at a suitable assumed solution.

Recently, renewed emphasis was placed on the Ritz - Galerkin approach to steady-state oscillations of nonlinear mechanical and electrical systems by the work of Klotter. Using an assumed solution in the form of a fundamental harmonic with undetermined amplitude and phase, Klotter [3, 4] treated single-degree-of-freedom systems with arbitrary, odd damping and restoring forces and obtained two nonlinear, algebraic equations in terms of the unknown quantities. These equations were then analyzed for specific cases and response curves obtained. The results were also compared to exact solutions in cases where exact solutions are available. In 1957, Cobb and Klotter [6] extended the use of the Galerkin method by employing a different form of assumed solution. Instead of using trigonometric functions, an approximation in the form of a parabola of undetermined degree was used. The coefficient of the solution was taken as a second undetermined parameter. The fact that the assumed solution was not periodic was accounted for by treating only the quarter-period and its periodic extension. This procedure was applied to single-degree-of-freedom, undamped systems with the excitation in the nonautonomous case, taken in the same form as the solution.

The following year, Harvey [31] obtained a relation for the ratio of the forcing function to the restoring force of a single-degree-of-freedom system, in terms of the periods of the free oscillation and the harmonic excitation. This treatment was restricted to what was called ''natural forcing functions'', that is, those which produce motion of the same form as the free oscillations. This restriction enabled the author to reduce the governing nonautonomous equation to an autonomous equation which can be more easily solved.

Research has also continued in the area of the topological methods. One such method was presented in 1932 by Meissner [7]. This method was unique among the geometrical approaches at that time, in that it is capable of handling nonautonomous systems. Previously, forced vibrations could be treated by graphical methods only when the nonautonomous equation could be reduced to an autonomous one. In some cases, such a reduction can be accomplished by the stroboscopic method described by Minorsky [8].

In 1953, a method analogous to the phase plane method was presented by Ku [9]. He utilized curves of acceleration versus displacement for different constant values of velocity. This acceleration plane method, like Meissner's method, is also applicable to forced vibrations.

b) Multi-degree-of-freedom systems - Ku later extended his method to deal with systems of several degrees of freedom [10]. In this presentation, the n differential equations governing an n-degree-of-freedom system are transformed into an n-th order differential equation and then the space trajectories of the higher order equation are analyzed. Collatz [11] has pointed out that analytical complications may arise in some cases, rendering such a transformation of a low order system to a smaller, higher order system, of little use.

At the time of Ku's work in geometrical methods, the analytical methods were also being extended into the realm of multi-degree-of-freedom problems. The Galerkin method was extended to nonlinear, two-degree-of-freedom, mechanical systems by Arnold [12] in 1953 and the following year, Klotter [13] demonstrated the application of the identical procedure to nonlinear, two-loop, conservative circuits. While the geometrical approach to multi-degree-of-freedom problems involves the analysis of trajectories in more than two dimensions, the difficulty with the analytical approach is the large number of nonlinear equations to be solved. As a result, most authors have limited their work to systems of two degrees of freedom.

The analytical study of nonlinear, multi-degree-of-freedom systems actually began to intensify with the consideration of nonlinear vibration absorbers. Linear absorbers were quite effective in eliminating the motion of the main mass provided the absorber spring and mass could be suitably adjusted with respect to the frequency of excitation. Should this frequency vary however, either by design or by accidental drift, such an adjustment would no longer be possible and furthermore, there would be the danger of the vibration being amplified if the driving frequency approached a natural frequency of the system. It was thus hoped that a nonlinear absorber spring would serve the purpose and overcome the difficulties.

In 1952, Roberson [23] analyzed an absorber system where the restoring force from the absorber spring consisted of a linear plus a cubic term. Using a single-term approximation obtained by Duffing's method, Roberson presented optimum design parameters as functions of a preassigned operating frequency.

The following year, Pipes [14] analyzed such a two-degree-of-freedom system where the nonlinear absorber spring had a hyperbolic sine spring characteristic. The equations of motion were combined into a fourth order, nonlinear equation and a first approximation to its solution was obtained by the Duffing method in terms of modified Bessel functions of the first kind. The amplitude-frequency relation was then solved graphically.

Shortly thereafter, Arnold [15] disputed a result obtained by Pipes. The discrepency concerned the occurrence of resonance peaks, that is infinite response amplitudes at finite amplitudes and frequencies of the forcing function. Pipes found no such resonance peaks to exist within the frequency ranges considered. Arnold used the Ritz - Galerkin method to investigate many combinations of hardening, softening and linear springs, both with and without the presence of forcing functions. He found that the nonlinear system can have, at most, one resonance peak provided one of the springs is linear and the force is applied to the main mass. In particular, if the coupling or absorber spring alone is nonlinear, as in Pipes' system, then the resonant frequency is equal to that of a single-degree-of-freedom system consisting of the linear main spring and a mass equal to the total mass of the two-degree-of-freedom system. If, on the other hand, the main spring alone is nonlinear, Arnold found the resonant frequency to be that of the linear auxiliary system as a single oscillator.

Sethna [16] in 1955, considered the effects of small viscous damping in systems of one and two-degrees-of-freedom by the Duffing approach and compared his results with those obtained by analog computer. For the single-degree-of-freedom case, he obtained a second approximation to the solution, but for the two-degree-of-freedom problem, only a one term, first approximation was presented and thus, the problem was considerably simplified.

In 1955, Huang [24] applied the perturbation approach to the harmonic oscillations of two-degree-of-freedom systems. Huang considered several cases including free and forced, damped and undamped systems, all with one linear and one ''Duffing type'' spring. He also presented the frequency equations in graphical form (response curves or resonance curves) for some numerical examples. Third order subharmonic response was treated by the same author [25] by a Fourier Series approach where the undetermined coefficients of the series are solved, as in Stoker's treatment [26] of the analogous single-degree-of-freedom problem, by iteration. Rosenberg [28] later discussed the possible advantages of treating problems such as Huang's by iteration rather than perturbation.

Several years later, Duffing's iteration method was applied to undamped, nonlinear systems with three degrees of freedom and amplitude-frequency equations were presented [32].

A study of damped, nonlinear, two-degree-of-freedom systems and their governing differential equations was carried out from a mathematical point of view by Pinney [29, 30]. Pinney's method revealed four distinct ranges of response containing integral multiples and sub-multiples of the frequency of excitation. These ranges are determined by the relative magnitudes of the two linear natural frequencies and the driving frequency.

In addition to the study of the steady-state response of multi-degree-of-freedom systems, a great deal of work has been done in extending and generalizing such concepts as natural frequency, resonance, eigenvalues and normal modes, which had traditionally been associated solely with linear systems, to have significance in nonlinear systems as well. According to the more general definitions, these terms reduce to the familiar, linear phenomina as a special case. In 1961, Henry and Tobias [33] investigated the behavior of the so-called normal modes of a free, undamped, nonlinear system of two degrees of freedom. They transformed the equations of motion into non-dimensional, normal coordinates and a normal mode can then be defined as motion which can be described in terms of a single normal coordinate. The transformed equations in general, remain coupled in the nonlinear terms and the authors presented criteria for the uncoupling of the equations and the existence of motion in a single mode with no interference from the others. The stability of such modes at rest was also discussed and experiments were described, verifying the single mode motions.

Bycroft [34] analyzed two-degree-of-freedom systems governed by the same transformed but coupled equations, by the Lighthill-Poincaré perturbation technique. This method is described by Tsien [35] and as is typical of perturbation methods, the computations for the second and higher order terms of the series become increasingly complicated.

The normal modes of nonlinear, multi-degree-of-freedom systems and their stability have also been studied from topological considerations, principally by Rosenberg and his associates, whose work has recently been consolidated into one publication [27].

4. Numerical Methods

In the absence of exact, closed-form solutions to nonlinear differential equations, the approximate methods previously discussed are a useful and valuable means of studying nonlinear vibrations problems. Involved in the use of these methods however, are certain shortcomings, some of which have already been mentioned. The methods in general, take no account of initial conditions and do not yield transient response. Furthermore, in most of the analytical methods, the form of the solution is determined solely by the initial approximation or assumed solution. One therefore obtains as a solution no more than what is initially requested and many important components or features of the response may thus be overlooked. Finally, those methods which apply to multi-degree-of-freedom problems, become prohibitive when treating systems larger than two degrees of freedom particularly when damping and several different nonlinearities are present simultaneously.

The development of the digital computer has made it possible to treat large and complex systems with comparative ease by numerical methods. Initial value problems lend themselves particularly well to step-by-step, numerical integration techniques. In addition to treating initial conditions and yielding the total response, the numerical approach affords certain other advantages over the other approximate methods. A wider range of parameters can be handled with fewer restrictions on their magnitude. There is also the feasibility of treating variable coefficients and even time-variant quantities which cannot be expressed in functional form. The case of many independently varying forces also presents no complication in a numerical approach.

a) Discrete Systems - Long before the digital computer made practical the solution of complex problems of many degrees of freedom, numerical methods were being used in the solution of simpler problems. One of the most obvious means of obtaining a numerical solution to an initial value problem is by a Taylor Series expansion about the initial time. As early as 1890, a numerical scheme was devised by Störmer [36] to improve on the accuracy and reduce the amount of work involved in numerical solutions by Taylor Series.

In 1951, which in terms of the machines in use today, can still be considered as belonging to the pre-computer era, an iterative method for the undamped, periodic motion of nonlinear, single-degree-of-freedom systems was presented by Brock [37]. The procedure begins with an assumed solution defined in analytic or numerical form, over an undetermined quarter-period and satisfying prescribed conditions at the beginning and end of the quarter-period. A second approximation is obtained by twice integrating the equation of motion. The integration is generally but not necessarily carried out numerically. The second approximation is then made to satisfy the original initial and quarter-period conditions by appropriately adjusting the length of the quarter-period. This procedure is repeated until the solutions converge.

In yielding only periodic solutions and in requiring an initial assumption, Brock's approach involves two disadvantages of the analytic methods. A general, step-by-step numerical integration method due to Newmark [38] appeared in 1959. Newmark presented equations for the solution at the end of a small time interval in terms of the known solution at the beginning of the interval. The application of the method to a number of linear problems was described. Certain special cases of the method, which correspond to different simplifying assumptions for the variation of the acceleration over the small time interval, were discussed. These special cases, often called acceleration methods, can be obtained directly from Taylor Series expansions with specific higher order terms neglected [39]. The retention of devivatives up to order three, corresponds to the assumption of linear acceleration. Accuracy and convergence can be improved by using smaller time increments and by retaining more terms of the Taylor expansions, that is approximating the acceleration by second or higher degree curves.

In the application of the acceleration methods to multi-degree-of-freedom systems, Newmark's equations become systems of equations which are linear or nonlinear depending on the physical system. A discrete system having a finite number of degrees of freedom and equations of motion in time alone, can be treated directly by these methods. A continuous structure having infinite degrees of freedom and governed by partial differential equations in space and time, can be idealized as a discrete system by a lumped mass or finite element technique.

A modified linear acceleration procedure was proposed in 1966 by Chaudhury, Brotton and Merchant [40]. The method was compared to Newmark's linear acceleration method and to Timoshenko's constant acceleration method [41] for the case of the free vibrations of a simple harmonic oscillator of one degree of freedom and was found to be the most accurate.

The authors described the application of the method to nonlinear framed structures with the spacial coordinates handled by the stiffness method and the individual member stiffness matrices modified by stability coefficients to account for nonlinear effects. Graphical results were shown for a triangular frame, fixed at the base and with three degrees of freedom of the apex.

b) Continuous Systems - A continuous, nonlinear system need not necessarily be idealized by a discrete system in order to eliminate the spacial variables. It may be more convenient to deal with the partial differential equations directly, using a finite difference scheme or a Galerkin approach in the spacial variables to reduce the equations to an ordinary nonlinear system in time alone. It should perhaps be emphasized that the Galerkin Method referred to here involves solutions or modes in the spacial variables while the same method mentioned previously in connection with multi-degree-of-freedom discrete systems, involves solutions in the time variable. The Galerkin Method has generally been preferred to a completely numerical approach since the computations are shorter and the semi-analytic form of the solution may be preferable to a purely numerical form. In the case of a one or two-mode Galerkin approach, the resulting ordinary differential equations in time may be solved by any of the previously mentioned methods. When more than two modes are considered however, the only practical alternative is a numerical solution.

In recent years, this type of approach has been used to solve the nonlinear oscillations of several continuous physical systems. In 1963, Bolotin [42] solved the nonlinear oscillations of plates by a two-mode solution in the spacial variables and the method of Krylov - Bogoliubov in the time variable. In many cases, dynamic stability has also been investigated and the use of step-by-step, numerical integration in time has proven so powerful that static stability problems have also been treated as time-dependent processes. Feodosiev applied this approach to the static snapping of shallow spherical caps [43] and to the elasto-plastic buckling of compressed bars [44]. In 1966, Dowell [45] considered nonlinear oscillating plates by a higher mode Galerkin approximation than that used by Bolotin and consequently, the system of time equations was then integrated numerically. The flexural vibrations of thin, circular rings were studied by Evensen [48] in 1965 by the same general approach. Using a two-term solution in space, Evensen solved the resulting ordinary differential equations by the method of Krylov and Bogoliubov.

The Galerkin method in the spacial coordinates has also been applied to the dynamic stability of shallow arches. In 1953, Hoff and Bruce [46] investigated the effects of the rise of the arch on the equilibrium configurations. Several types of pressure loads were considered and the study was purely analytic.

In 1966, a numerical study of the same system under step pressure loads was conducted by Lock [47]. Using a two-mode analysis, critical pressures were obtained and the effects of viscous damping were demonstrated. The rise of the arch was shown to influence the response of the second mode and in turn, the stability of the structure.

IV. FORMULATION OF THE PROBLEMS

1. Discrete System

a) Assumptions - A discrete, mass-spring system having a finite number of degrees of freedom is shown schematically in Fig. 1. It is presumed that forces act at mass points only. For the sake of simplicity, only adjacent masses are coupled and only the Nth mass is grounded. Inter-coupling among all the masses does not affect the validity of the procedure but does considerably complicate the governing equations.

Any or all of the springs may be nonlinear and the nonlinearities may be unequal. The spring characteristics are assumed to be of a form such that the restoring force in any spring is an odd function of the extension of that spring. For the present, an odd polynominal will be assumed and only linear and cubic terms will be retained. It will subsequently be demonstrated that the procedure is also valid for the cases where the system is purely nonlinear and where the restoring forces are unsymmetrical.

The loads $P_n(t)$ are arbitrary functions and may begin or cease to act at any prescribed time. Any of the loads may be equal to zero and as may be expected, all the loads may vanish provided that one non-zero initial condition is prescribed. The step-by-step numerical integration procedure makes it possible to treat loads which are not expressed in functional form and thus experimental results in numerical form may be handled. The nature of the method also makes feasible the solution of variable coefficient equations which constitute a difficult enough problem in linear theory.

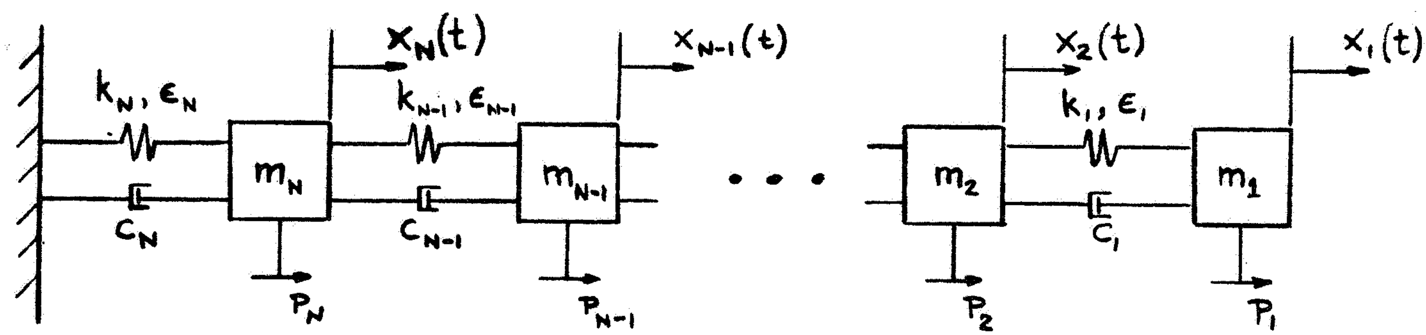

FIG. 1 — DISCRETE SYSTEM

b) Equations - From Newton's second law or d'Alembert's Principle, the system shown in Figure 1 is found to be governed by the following set of N equations for the displacements $x_n(t)$

$$m_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) + \epsilon_1(x_1 - x_2)^3 - P_1(t) = 0$$

$$\begin{aligned} m_n\ddot{x}_n &- c_{n-1}(\dot{x}_{n-1} - \dot{x}_n) - k_{n-1}(x_{n-1} - x_n) - \epsilon_{n-1}(x_{n-1} - x_n)^3 \\ &+ c_n(\dot{x}_n - \dot{x}_{n+1}) + k_n(x_n - x_{n+1}) + \epsilon_n(x_n - x_{n+1})^3 - P_n(t) = 0 \\ &\qquad n = 2, \ldots, N-1 \end{aligned} \tag{1}$$

$$\begin{aligned} m_N\ddot{x}_N &- c_{N-1}(\dot{x}_{N-1} - \dot{x}_N) - k_{N-1}(x_{X-1} - x_N) - \epsilon_{N-1}(x_{N-1} - x_N)^3 \\ &+ c_N\dot{x}_N + k_N x_N + \epsilon_N x_N^3 - P_N(t) = 0 \end{aligned}$$

with the set of initial conditions arbitrarily prescribed as

$$x_n(t_i) = x_{ni}$$

$$\dot{x}_n(t_i) = \dot{x}_{ni} \qquad n = 1, \ldots, N$$

where the dot denotes differentiation with respect to time. The subscript i is used throughout to designate "initial" quantities, that is those evaluated at $t = t_i$, the beginning of the time interval Δt. Similarly, the subscript f denotes "final" quantities, evaluated at $t = t_f = t_i + \Delta t$.

Having prescribed all displacements and velocities at $t = t_i$, the initial accelerations can be computed from Eqs. (1). With the complete solution determined at $t = t_i$, the solution at $t = t_f$ is obtained by assuming a linear variation of acceleration over the time interval. This assumption yields a special case of the equations presented by Newmark (38) for the accelerations and velocities at $t = t_f$

$$\ddot{x}_{nf} = \frac{6}{\Delta t^2} \left[x_{nf} - A_n \right] \tag{2a}$$

$$\dot{x}_{nf} = \frac{\Delta t}{2} \ddot{x}_{nf} + \frac{B_n}{\Delta t} \tag{2b}$$

where

$$A_n = x_{ni} + (\Delta t) \dot{x}_{ni} + \frac{(\Delta t)^2}{3} \ddot{x}_{ni}$$

and

$$B_n = \Delta t \left(\dot{x}_{ni} + \frac{\Delta t}{2} \ddot{x}_{ni}\right)$$

$$n = 1, \ldots, N$$

Eqs. (1) written at $t = t_f$ together with Eqs. (2a) and (2b) constitute three systems of equations for the three vector quantities x_{nf}, $\dot{x}_{nf}$ and $\ddot{x}_{nf}$ in terms of the known vectors at $t = t_i$. By eliminating the accelerations and velocities in Eqs. (1), (2a) and (2b), the following system of N algebraic equations for the N displacements $x_n(t_f)$ is obtained

$$
\left.
\begin{aligned}
&\left[S_1 + R_1 + k_1\right] x_{1f} - \left[R_1 + k_1\right] x_{2f} + \epsilon_1 (x_{1f} - x_{2f})^3 \\
&\qquad = S_1 A_1 + R_1 \left[(A_1 - A_2) - \left(\frac{B_1 - B_2}{3}\right)\right] + P_1 (t_f) \\
&- \left[R_{n-1} + k_{n-1}\right] x_{n-1,\,f} + \left[S_n + R_{n-1} + R_n + k_{n-1} + k_n\right] x_{nf} \\
&- \left[R_n + k_n\right] x_{n+1,\,f} - \epsilon_{n-1} (x_{n-1,\,f} - x_{nf})^3 + \epsilon_n (x_{nf} - x_{n+1,\,f})^3 \\
&\qquad = S_n A_n - R_{n-1} \left[(A_{n-1} - A_n) - \left(\frac{B_{n-1} - B_n}{3}\right)\right] \\
&\qquad + R_n \left[(A_n - A_{n+1}) - \left(\frac{B_n - B_{n+1}}{3}\right)\right] + P_n (t_f) \\
&\qquad\qquad n = 2, \ldots, N-1 \\
&- \left[R_{N-1} + k_{N-1}\right] x_{N-1} + \left[S_N + R_{N-1} + R_N + k_{N-1} + k_N\right] x_{Nf} \\
&- \epsilon_{N-1} (x_{N-1,\,f} - x_{Nf})^3 + \epsilon_N x_{Nf}^3 \\
&\qquad = S_N A_N - R_{N-1} \left[(A_{N-1} - A_N) - \left(\frac{B_{N-1} - B_N}{3}\right)\right] \\
&\qquad + R_N \left[A_N - \frac{B_N}{3}\right] + P_N (t_f)
\end{aligned}
\right\} \quad (3)
$$

where R_n and S_n are system constants defined as

$$
R_N = \frac{3c_n}{\Delta t} \quad \text{and} \quad S_n = \frac{6m_n}{(\Delta t)^2} \quad n = 1, \ldots, N.
$$

Solving Eqs. (3) for the displacements, the accelerations and velocities are obtained from Eqs. (2a) and (2b) respectively. The entire procedure can then be repeated for the next time interval, beginning with the solution at $t = t_f$.

The preceeding equations are developed in greater detail in Appendix B. The system of nonlinear algebraic equations (3) is solved by the method of continuity which is described in Appendix C.

For the case where the equations of motion, Eqs. (1), are purely non-linear, that is when the coefficients k_n of the linear terms vanish, it can be seen that the coefficients of the displacements x_{nf} in Eqs. (3) remain non-zero.

Furthermore, the right side of Eqs. (3) consists of the loads and of the A_n and B_n which are functions of the known solution at $t = t_i$. Thus the Eqs. (3) remain non-homogeneous even when the loads $P_n(t)$ all vanish identically, provided one non-zero initial condition exists.

2. Continuous System - Shallow Arch

a) Assumptions - A simply supported, shallow arch acted on by a time dependent, distributed load $p(x, t)$ is shown in Fig. 2. All forces and displacements are shown in the positive sense, consistent with the adopted sign convention. The supports are a fixed distance L, apart and the axial load P, resulting from the stretching of the middle line, is assumed to be constant in x. The shape of the unstressed member is described by the curve $z = \overline{w}(x)$

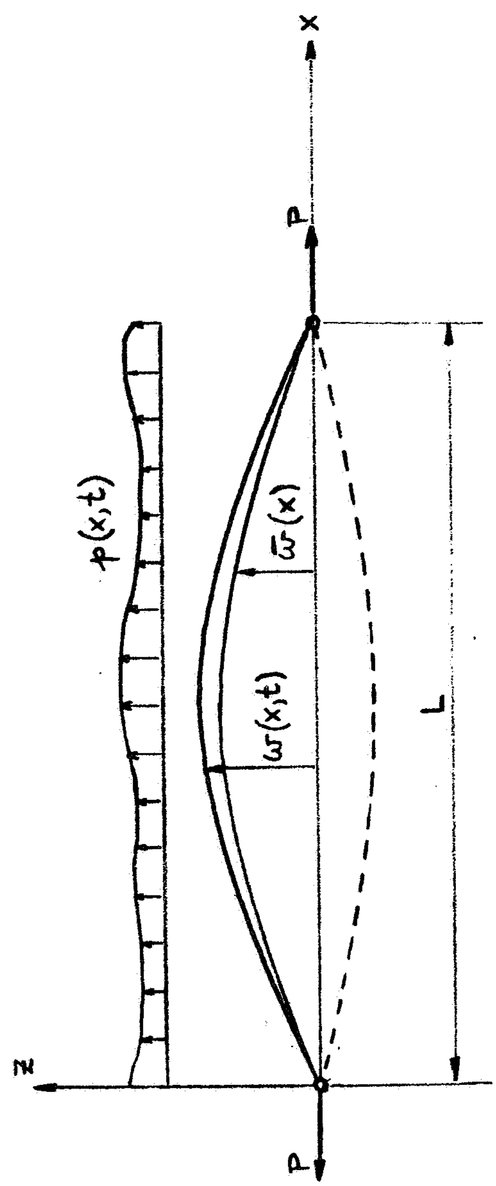

FIG. 2 — SHALLOW ARCH

and the deflected position is $z = w(x, t)$. All the usual assumptions as to material and section properties, of small deflection, elastic beam theory, are retained. Thus strains are assumed to be negligible compared to unity and to be below the elastic limit of the material where Hooke's Law is assumed valid. The cross-section is uniform throughout the length and one principal axis is in the plane of bending. Rotatory inertia and shear deformations are neglected.

b) Equation of Motion - The partial differential equation governing the motion of the arch shown in Fig. 2 has been derived by several authors. Marguerre (51) derived the equation from energy principles by the calculus of variations while Biezeno (52) and Timoshenko (53) whose approach is followed here, derived the equation from the moment-curvature relationship.

With the assumptions just stated, the equation of the elastic curve $w - \overline{w}$, may be written

$$\frac{\partial^2}{\partial x^2}(w - \overline{w}) = \frac{M}{EI}$$

where E is Young's Modulus and I is the moment of inertia of the cross-section. The moment M is taken as positive when it produces tension in the lower fibers of the cross-section. Substituting for the moment, the equation can be rewritten as

$$EI \frac{\partial^2}{\partial x^2}(w - \overline{w}) = Pw - M\left[p(x, t)\right]$$

where

$$P = \frac{AE}{2L} \int_0^L \left[\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{d\bar{w}}{dx}\right)^2 \right] dx$$

and M(p) is the moment due to the dynamic load p(x, t). Differentiating twice with respect to x yields

$$EI \left(\frac{\partial^4 w}{\partial x^4} - \frac{d^4 \bar{w}}{dx^4} \right) - P \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 M(p)}{\partial x^2} = 0$$

Considering, according to d'Alembert's Principle, the inertial and damping forces acting to oppose motion, we have,

$$\frac{\partial^2 M(p)}{\partial x^2} = - p(x, t) + \rho A \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t}$$

where ρ is the volume mass density, A is the cross-sectional area and c is a damping coefficient. Thus, the equation of motion, with P as previously defined, now becomes

$$EI \left(\frac{\partial^4 w}{\partial x^4} - \frac{d^4 \bar{w}}{dx^4} \right) - \frac{AE}{2L} \frac{\partial^2 w}{\partial x^2} \int_0^L \left[\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{d\bar{w}}{dx}\right)^2 \right] dx$$

$$+ \rho A \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} - p(x, t) = 0. \qquad (4)$$

By introducing the independent variables

$$\xi = \frac{\pi x}{L} \quad \text{and} \quad \tau = \omega_o t$$

where

$$\omega_0 = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

the Eq. (4) can be expressed nondimensionally as

$$\frac{\partial^4 \eta}{\partial \xi^4} - \frac{d^4 \bar{\eta}}{d\xi^4} - \frac{1}{2\pi} \frac{\partial^2 \eta}{\partial \xi^2} \int_o^{\pi} \left[\left(\frac{\partial \eta}{\partial \xi}\right)^2 - \left(\frac{d\bar{\eta}}{d\xi}\right)^2 \right] d\xi$$
$$+ \frac{\partial^2}{\partial \tau^2} + \gamma \frac{\partial \eta}{\partial \tau} - q(\xi, \tau) = 0 \tag{5}$$

where

$$\eta(\xi, \tau) = w/r$$

r being the radius of gyration of the section. The unstressed position of the arch is described by $\bar{\eta}$ and the nondimensional load and damping coefficient are

$$q(\xi, \tau) = \left(\frac{L}{\pi}\right)^4 \frac{p(x, t)}{EIr} \quad \text{and} \quad \gamma = \frac{c}{\rho A \omega_0} .$$

For simple supports, the boundary conditions on Eq. (5) are

$$\eta(0, \tau) = \eta(\pi, \tau) = \frac{\partial^2 \eta(0, \tau)}{\partial \xi^2} = \frac{\partial^2 \eta(\pi, \tau)}{\partial \xi^2} = 0.$$

The initial conditions are arbitrarily prescribed as

$$\eta(\xi, \tau_i) = \eta_i(\xi) \quad \text{and} \quad \dot{\eta}(\xi, \tau_i) = \dot{\eta}_i(\xi)$$

where the dot denotes differentiation with respect to τ.

c) Galerkin's Method in the Spacial Coordinates - The partial differential equation (5) together with the boundary and initial conditions, can be solved by the numerical approach described in Section IV-1 if the spacial coordinates can be separated and an ordinary differential equation in time alone, obtained. This has been done in the case of the shallow arch (46, 47) and for a number of other problems (43, 44, 45, 48) by the application of Galerkin's Method in the spacial variables with the undetermined coefficients taken to be functions of time.

Thus, a suitable solution for the pin-ended arch shown in Fig. 2 is assumed to be

$$\eta(\xi, \tau) = \bar{\eta}(\xi) + \sum_{n=1}^{N} a_n(\tau) \sin n\xi \tag{6}$$

which satisfies the stated boundary conditions. The load, q, is similarly expanded as

$$q(\xi, \tau) = \sum_{n=1}^{N} q_n(\tau) \sin n\xi .$$

These expansions are now substituted into the Eq. (5) yielding an expression of the form

$$\sum_{n=1}^{N} f_n (a_1, \ldots, a_N) \sin n\xi = 0$$

For this equation to hold for all ξ in the interval $0 \leq \xi \leq \pi$, we require all the coefficients f_n to vanish. This results in a system of N ordinary differential equations for the N $a_n(\tau)$. If the undisplaced shape of the arch is taken to be sinusoidal, that is $\bar{\eta} = e \sin \xi$, the system of equations for the $a_n(\tau)$ becomes

$$\left.\begin{aligned} &\ddot{a}_1 + \gamma \dot{a}_1 + a_1 + \frac{1}{4} \left(\sum_{j=1}^{N} j^2 a_j^{\,2} + 2e\, a_1 \right) (a_1 + e) - q_1(\tau) = 0 \\ &\ddot{a}_n + \quad \dot{a}_n + n^4 a_n + \frac{n^2}{4} \left(\sum_{j=1}^{N} j^2 a_j^{\,2} + 2e\, a_1 \right) a_n - q_n(\tau) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad n = 2, \ldots, N \end{aligned}\right\} \quad (7)$$

The spacial variable has now been separated out of the equation and the system of infinite degrees of freedom, governed by a nonlinear, partial differential equation has been reduced to a system of N degrees of freedom governed by N nonlinear, ordinary differential equations. The boundary conditions on the problem have been satisfied by the assumed solution, Eq. (6), and the initial conditions are now prescribed for all the $a_n(\tau)$ as

$$a_n(\tau_i) = a_{ni} \quad \text{and} \quad \dot{a}_n(\tau_i) = \dot{a}_{ni}$$

$$n = 1, \ldots, N.$$

It can be seen from an examination of Eqs. (7) that any mode n, not driven by a load component q_n' of that mode, will not participate in the total response unless it is excited by a non-zero initial condition a_{ni} or $\dot{a}_{ni}$.

d) Solution by Linear Acceleration - With the problem now described by Eqs. (7) and the a_{ni} and $\dot{a}_{ni}$ prescribed, the solution a_{nf} can be obtained by the same numerical integration procedure which was used to solve Eqs. (1). From the initial conditions, the initial acceleration components $\ddot{a}_{ni}$ are obtained from Eqs. (7). Then, assuming the variation of the $\ddot{a}_n$ to be linear over the small interval $\Delta\tau$, the following nonlinear algebraic systems are obtained for the a_{nf}, $\ddot{a}_{nf}$ and $\dot{a}_{nf}$:

$$\left.\begin{aligned}
&(S^* + R^* + 1)\, a_{1f} + \frac{1}{4}\left(\sum_{j=1}^{N} j^2 a_{jf}^2 + 2e\, a_{1f}\right)(a_{1f} + e) \\
&\qquad = q_1(\tau_f) + S^* A_1 + R^*\left(A_1 - \frac{B_1}{3}\right) \\
&(S^* + R^* + n^4)\, a_{nf} + \frac{n^2}{4}\left(\sum_{j=1}^{N} j^2 a_{jf}^2 + 2e\, a_{1f}\right) a_{nf} \\
&\qquad = q_n(\tau_f) + S^* A_n + R^*\left(A_n - \frac{B_n}{3}\right) \\
&\qquad\qquad n = 2, \ldots, N.
\end{aligned}\right\} \tag{8}$$

$$\ddot{a}_{nf} = S^*(a_{nf} - A_n) \qquad n = 1, \ldots, N \tag{9}$$

$$\dot{a}_{nf} = \frac{\Delta\tau}{2}\,\ddot{a}_{nf} + \frac{Bn}{\Delta\tau} \qquad n = 1, \ldots, N. \tag{10}$$

The constants R* and S* are defined as

$$R^* = \frac{3\gamma}{\Delta\tau} \qquad \text{and} \qquad S^* = \frac{6}{(\Delta\tau)^2}$$

and, as in Section IV-1, the A_n and B_n are known functions of the solution at $\tau = \tau_i$. Eqs. (8), (9) and (10) solved in sequence, yield the complete solution at any time τ_f in terms of the Fourier components a_n, $\dot{a}_n$ and $\ddot{a}_n$. The non-dimensional displacement, velocity and acceleration of the arch can then be obtained from the series expressed in Eq. (6) and its first and second partial derivatives with respect to τ. These quantities are in turn related to the dimensional solution by the expressions

$$w = r\eta$$

$$\frac{\partial w}{\partial t} = r\omega_0 \frac{\partial \eta}{\partial \tau}$$

$$\frac{\partial^2 w}{\partial t^2} = r\omega_0^2 \frac{\partial^2 \eta}{\partial \tau^2}$$

V. NUMERICAL STUDIES

A limited number of numerical examples are presented here, to illustrate the application of the procedure and to compare the results with those obtained by other methods. Many of the problems used for comparison are, of necessity, relatively simple to permit their solution by analytic methods. A few more complicated problems are solved to demonstrate the feasibility of the numerical approach.

The nonlinear algebraic systems which result from the numerical integration technique, are solved in all cases, by the "method of continuity" which is described in Appendix C.

The results presented herein were obtained by using the CDC 6600 electronic digital computer located at the Courant Institute of Mathematical Sciences of New York University.

Several particular cases of the general, discrete system shown in Fig. 1 are solved. Response curves for a system governed by the equation

$$3\,\ddot{x} + 0.25\,\dot{x} + 300\,x + \epsilon x^3 = 2\,\sin\omega t \tag{11}$$

are shown in Fig. 3 along with the curve for the adjacent linear problem. The amplitude-frequency relations for the systems with $\epsilon = 1$ and $\epsilon = 2$ were obtained by the perturbation method. The nonlinear response curves are seen to bend to the right of the linear problem. This is characteristic of

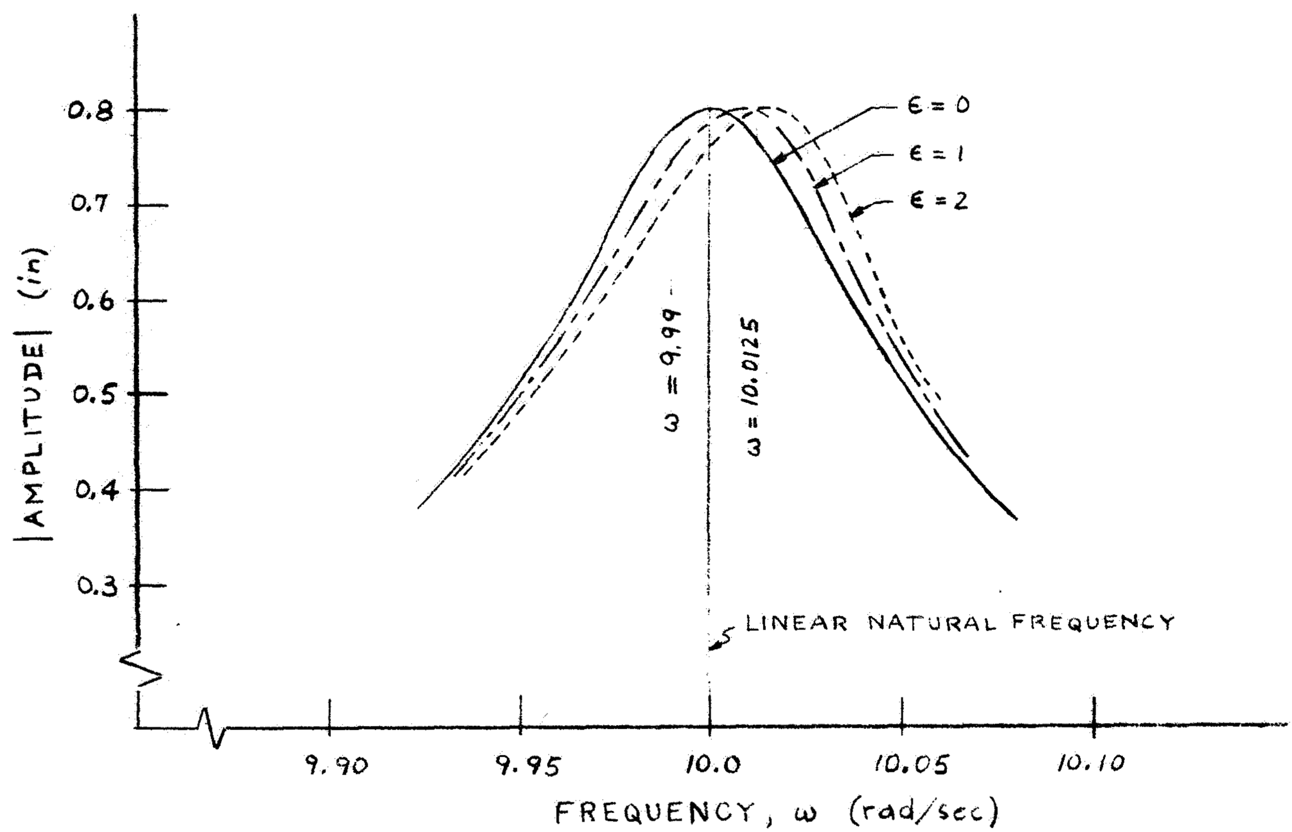

FIG. 3 – RESPONSE CURVES, HARD SPRING

such problems with "hardening" or positive nonlinearities. Response curves for the same problems but with "softening" or negative nonlinearities, would bend to the left.

The numerical values were chosen such that the nonlinear problems are far enough away from the linear problem to be easily distinguished and close enough for the perturbation method to be valid. The driving frequencies at which the numerical solution is to be compared to the perturbation solution, were chosen close to the linear natural frequencies of the system so that the three steady-state amplitudes are as distinct as possible. Driving frequencies both larger and smaller than the natural frequency were selected to examine the effect of the phase shift between the force and the response on the behavior of the systems.

In Fig. 4, the two nonlinear problems described by Eq. (11) with $\epsilon = 1$ and $\epsilon = 2$ are shown with $\omega = 9.99$. With zero initial conditions prescribed, the amplitudes are seen to approach their respective steady-state values as the transients in the system damp out. For the case where $\epsilon = 1$, the amplitudes reach the value of the steady-state amplitude in 70 seconds. The problem with $\epsilon = 2$ reaches the steady-state amplitude in 60 seconds. Therefore, if a system is known to be loaded for shorter periods of time, the steady-state solution could be an overly conservative design criterion.

Furthermore, it can be seen that the amplitudes actually exceed the predicted steady-state values so that the steady-state amplitude cannot always

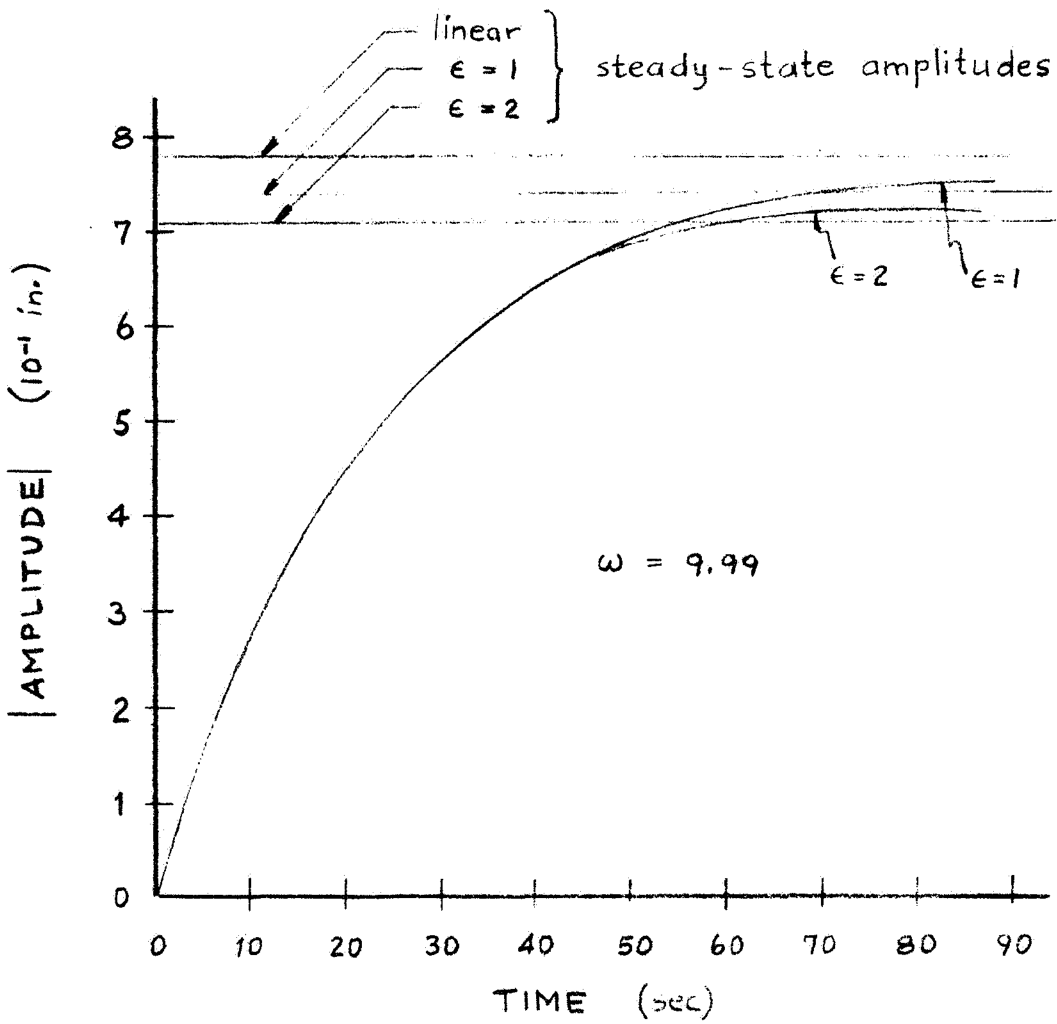

FIG. 4 – HARD SPRING, FORCED VIBRATION, $\omega < \omega_0$

be considered to be a limiting value. From Fig. 4, it appears that for larger values of time, the amplitudes may exhibit a damped oscillation about the steady-state value.

The amplitude growth for the same problem with $\omega = 10.0125$ is shown in Fig. 5 for the linear case and for the case of $\epsilon = 2$. The amplitudes appear to approach the steady-state values asymptotically here as opposed to the case shown in Fig. 4 where ω is less than the linear natural frequency.

It should be noted that the damping coefficient in Eq. (11) was deliberately chosen large so that the numerical solution would approach the predicted steady-state amplitudes as rapidly as possible. If the damping were small, as is often the case, both the steady-state amplitude and the time required for the transients to damp out would be greatly increased. If damping were neglected entirely, the transients would remain in the system and the response amplitudes would vary only because of phase differences between the transient and steady-state components. Thus, for oscillatory systems where the duration of loading is known to be limited and where damping is small, the steady-state solutions may be of less significance than indicated by the foregoing examples.

The free vibration of an undamped system with a softening nonlinearity is shown in Fig. 6. The system is described by the equation

$$3\ddot{x} + 300\,x - 2x^3 = 0$$

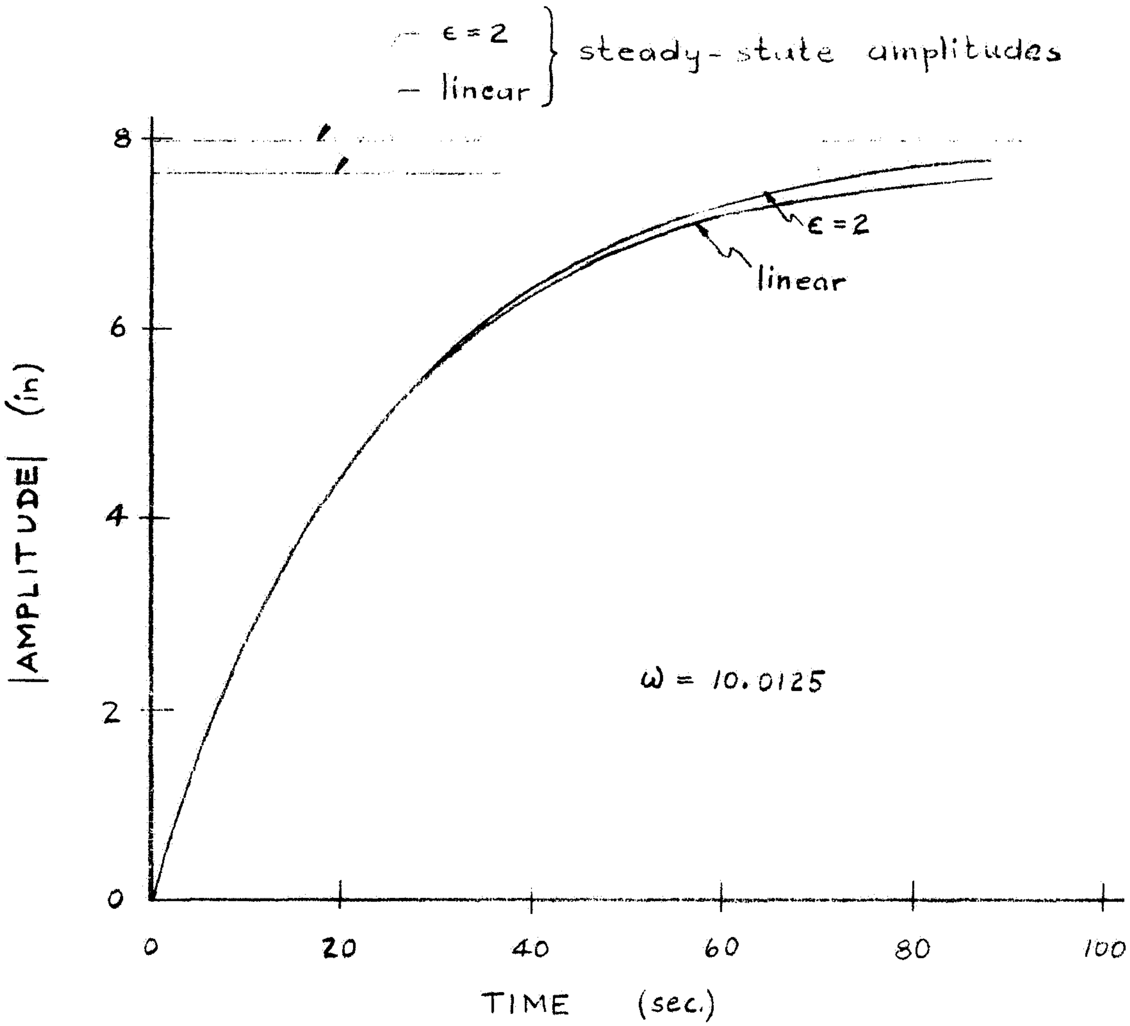

FIG. 5 – HARD SPRING, FORCED VIBRATION, $\omega > \omega_0$

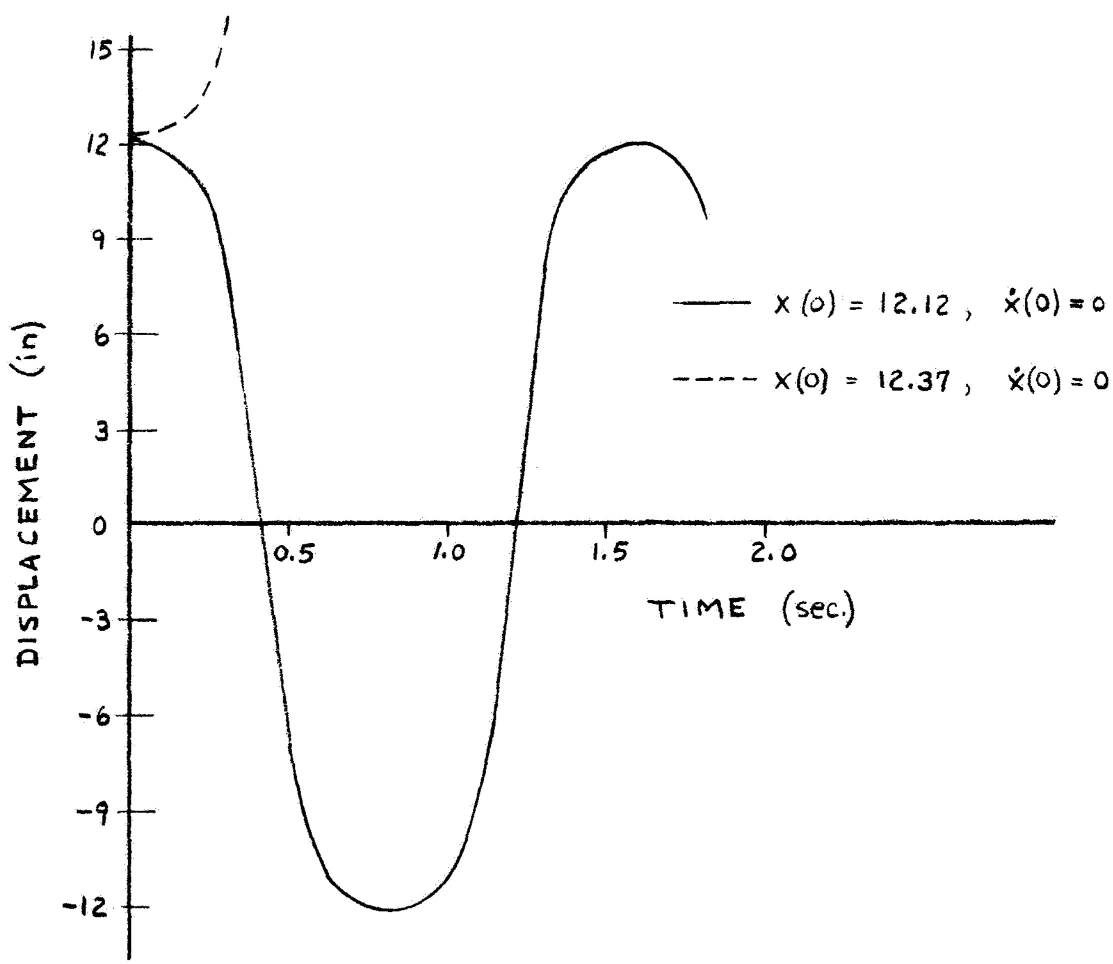

FIG. 6 - SOFT SPRING, INITIAL DISPLACEMENTS

A phase plane analysis of this system reveals a region of periodic oscillations about a stable equilibrium state [26]. Outside of this region, the solutions are not periodic. The transition curve separating these regions in the x-$\dot{x}$ plane for this problem, passes through the points $(\sqrt{150}, 0)$ and $(0, 50\sqrt{3})$. The solid curve in Fig. 6 represents the periodic response of the system when initialized by the conditions $x(0) = 0.99\sqrt{150}$ and $\dot{x}(0) = 0$. The initial conditions $x(0) = 1.01\sqrt{150}$ and $\dot{x}(0) = 0$ result in the nonperiodic motion shown by the dashed line.

As has already been mentioned, in the case of a softening spring the A-ω curve bends to the left of the curve for the linear problem. It can thus be expected that regardless of the amplitude chosen, the frequency of the free, nonlinear problem will be smaller than the linear natural frequency. For this problem, the linear natural frequency is 10 rad/sec. and the frequency of the softening system is approximately 3.93 rad/sec. at an amplitude of 12.1 in.

Fig. 7 shows the response of the same, soft spring system to initial velocities and zero initial displacements. The periodic response to an initial velocity of 85.7 in./sec. or slightly less than the critical value of $50\sqrt{3}$, is shown by the solid curve. An initial velocity of 87.5 in./sec. which is slightly larger than the critical velocity, results in the nonperiodic response shown by the dashed line.

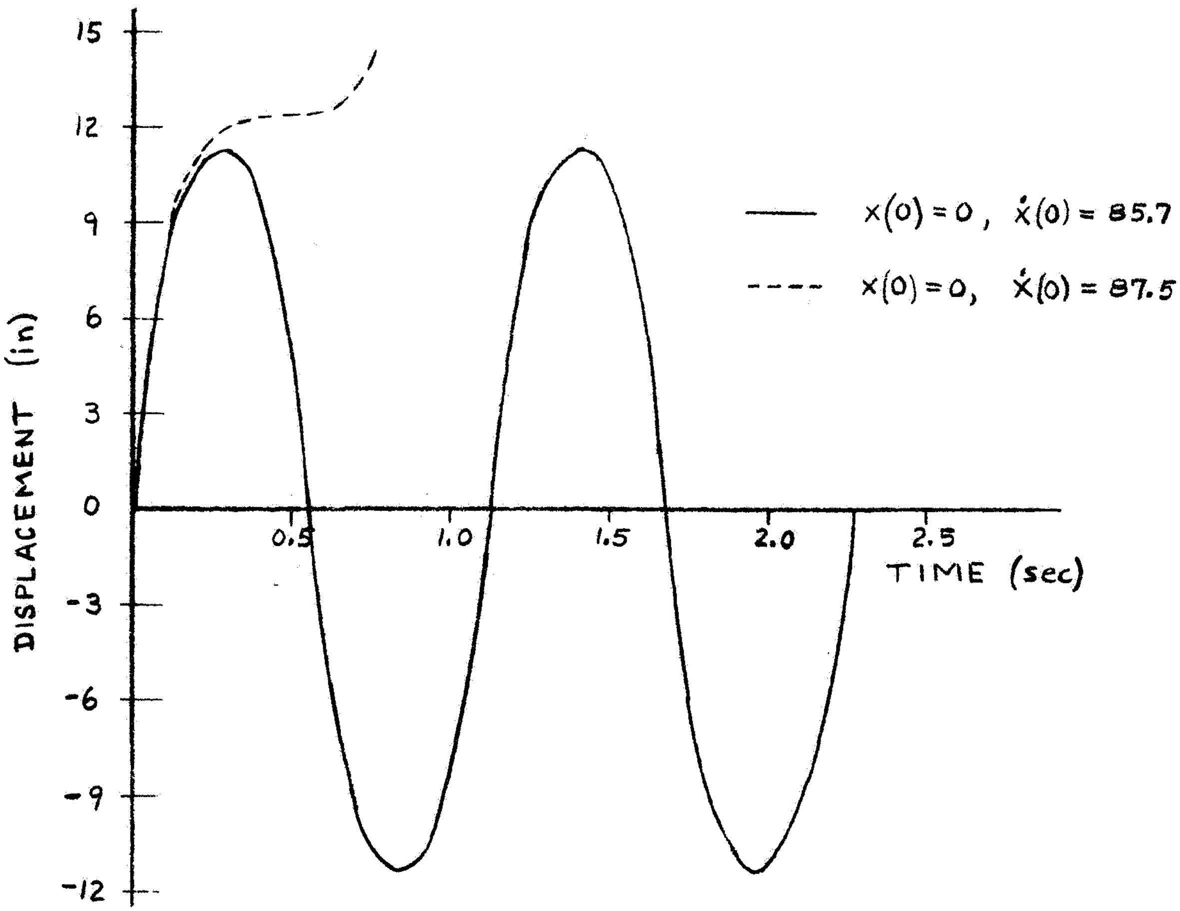

FIG. 7 - SOFT SPRING, INITIAL VELOCITIES

The displacements represented by the dashed lines in Figs. 6 and 7 increase without bound with increasing time. The initial displacement problem shown in Fig. 6 and the initial velocity problem shown in Fig. 7 may both be termed unstable in the sense that two initially adjacent systems do not remain in the same neighborhood.

The frequency of the periodic response in Fig. 7 is seen to be approximately 5.61 rad./sec. at an amplitude of 11.3 in. The difference in response frequencies between the two free vibrations shown in Figs. 6 and 7 points up a unique feature of nonlinear systems. Unlike the linear case where all free oscillations occur at the same frequency, in the nonlinear problem the amplitude of the free vibration determines its frequency.

The forced response of two systems with cubic nonlinearity and no linear term, is shown in Fig. 8. The dashed line represents an undamped system governed by the equation

$$\ddot{x} + x^3 = 2 \cos t \tag{13}$$

and the solid line represents a similar system with damping, described by the equation

$$\ddot{x} + 0.2\dot{x} + x^3 = 3 \cos t \tag{14}$$

the steady-state amplitude of the damped system is 0.3164 in. and due to the relatively large damping in the system, the transients are seen to damp out

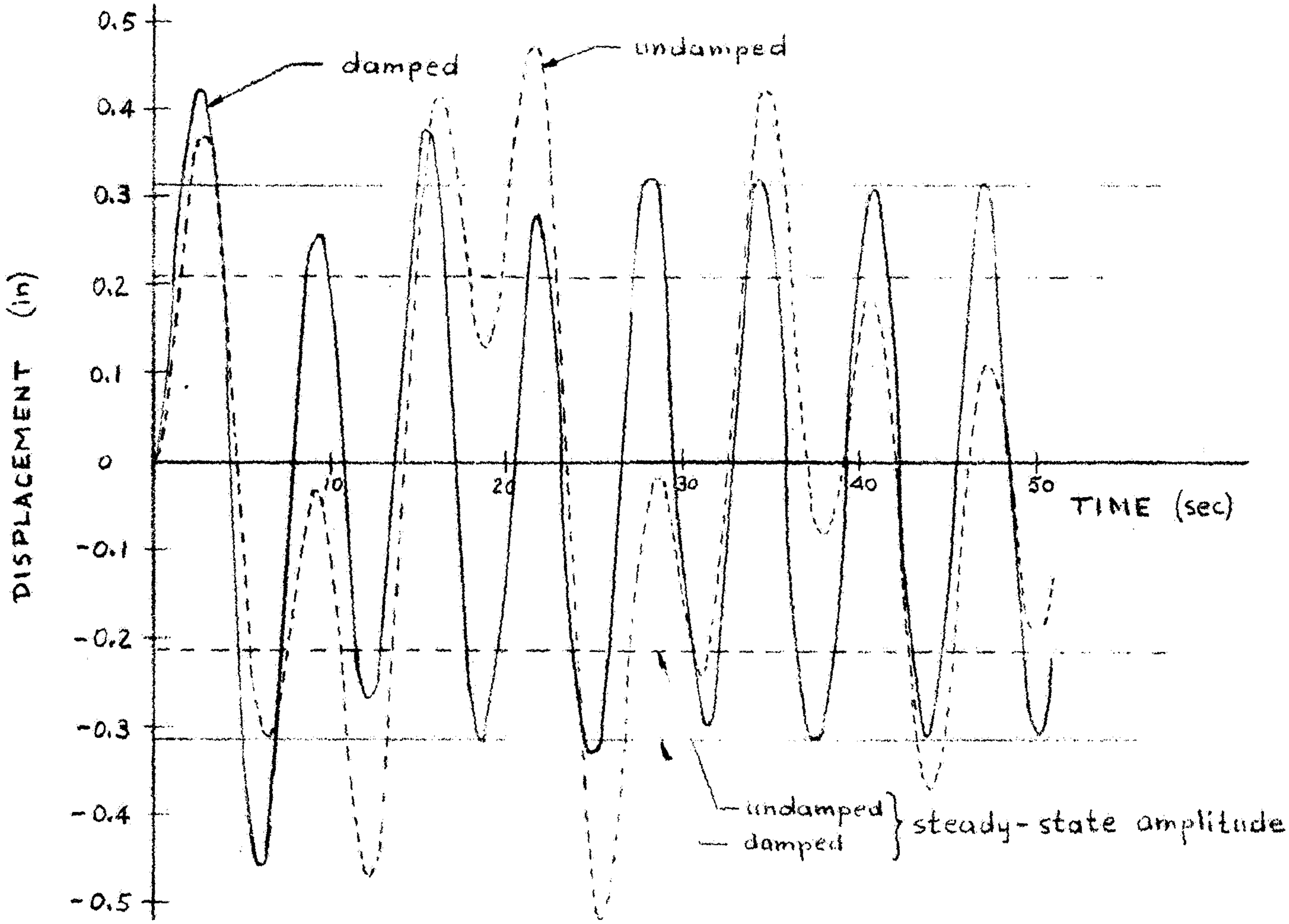

FIG. 8 — FORCED VIBRATION, NO LINEAR TERM

quite rapidly. By a topological analysis of this problem, it can be shown (5) that another set of initial conditions may yield a response approaching a different stable periodic solution.

The steady-state amplitude of the undamped problem is 0.207 in. and the presence of undamped transients is evidenced by the continued erratic pattern of the response.

All the single-degree-of-freedom systems considered thus far, have had restoring forces which were odd functions of the spring extensions. The addition of even-order terms into the general equations of motion creates no new problems with regard to the numerical procedure. The presentation of a single-degree-of-freedom system with a quadratic term included would be redundant however, because the quadratic term can be eliminated by a simple transformation. The transformed system is driven by a different forcing function but since the loading is completely arbitrary in the numerical procedure, the problem is essentially the same as that described by Eq. (11). For larger systems, such a transformation may not be possible however, and the solution of the original equations can then be obtained directly.

The amplitude growth with time for both masses of a nonlinear system of two degrees of freedom is shown in Fig. 9. The controlling differential equations can be obtained by setting $N = 2$ in Eq. (1) which yields

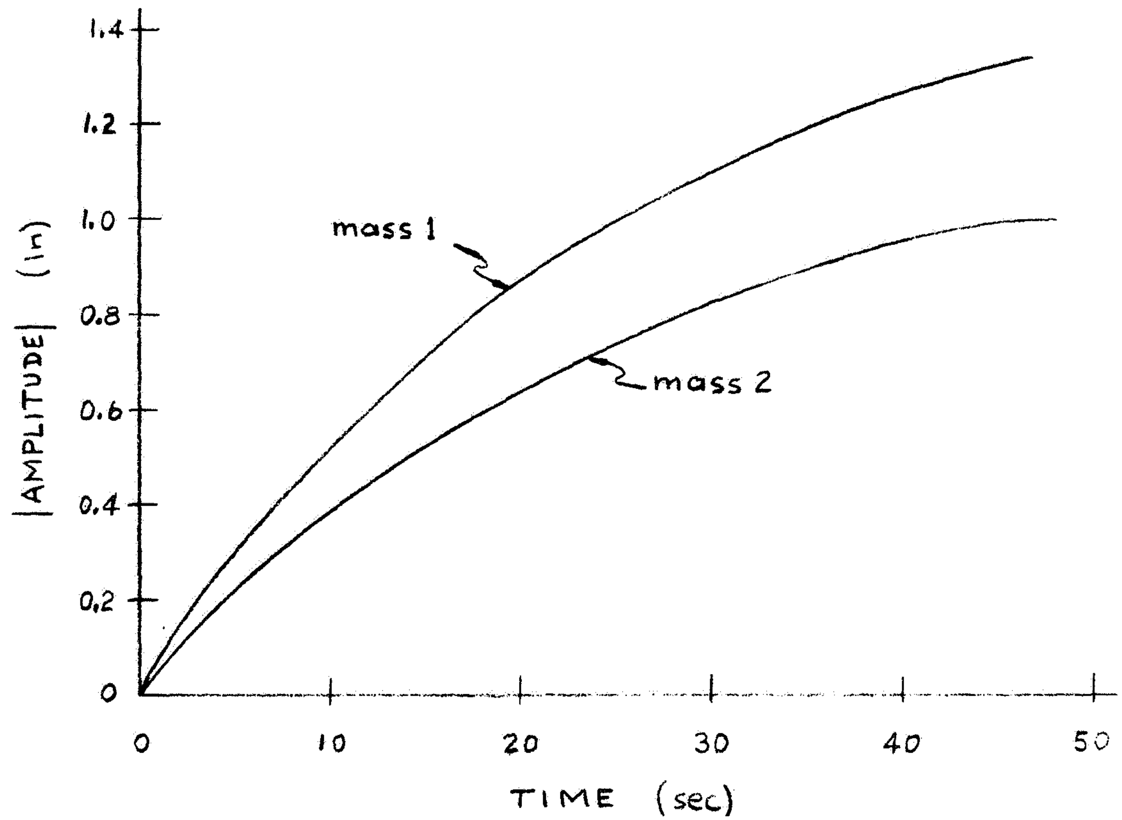

FIG. 9 – FORCED VIBRATION, 2 DEGREES OF FREEDOM

$$\left.\begin{aligned} &m_1 \ddot{x}_1 + c_1 (\dot{x}_1 - \dot{x}_2) + k_1 (x_1 - x_2) + \epsilon_1 (x_1 - x_2)^3 - P_1(t) = 0 \\ &m_2 \ddot{x}_2 - c_1 (\dot{x}_1 - \dot{x}_2) - k_1 (x_1 - x_2) - \epsilon_1 (x_1 - x_2)^3 \\ &\qquad + c_1 \dot{x}_2 + k_2 x_2 + \epsilon_2 x_2^3 - P_2(t) = 0 \end{aligned}\right\} \quad (15)$$

The choice of the numerical values used in the above equations was somewhat arbitrary but, as in the single-degree-of-freedom problem, it was sought to show large displacements and to have the transients damp out rapidly. Substituting the values used, the equations become

$$6 \ddot{x}_1 + (\dot{x}_1 - \dot{x}_2) + 600 (x_1 - x_2) + 2 (x_1 - x_2)^3 = 5 \sin 5.175\, t$$

$$\begin{aligned} &3 \ddot{x}_2 - (\dot{x}_1 - \dot{x}_2) - 600 (x_1 - x_2) - 2 (x_1 - x_2)^3 \\ &\qquad + \dot{x}_2 + 300 x_2 = 0 \end{aligned}$$

It should be noted that the fact that $c_1 = c_2$, $\epsilon_2 = 0$, and $P_2 = 0$ in Eq. (15) is purely arbitrary and irrelevant as far as the complexity of the numerical procedure is concerned.

It can be seen from Fig. 9 that the amplitudes of both masses appear to be approaching distinct, steady-state amplitudes in the same way as the single-degree-of-freedom systems shown in Figs. 4 and 5. No perturbation solution is available for comparison in this case, because the frequency equations are complicated by the inclusion of two phase angles to account for the damping in the system.

The response of masses 1, 2 and 3 of a three-degree-of-freedom system to a sinusoidal load applied to mass 1, is shown in Fig. 10. The differential equations of the system are obtained by setting N = 3 in Eq. (1). With the numerical values used, these equations become

$$\ddot{x}_1 + 0.01\,(\dot{x}_1 - \dot{x}_2) + 2\,(x_1 - x_2) + (x_1 - x_2)^3 = 0.4 \sin 10\,t$$

$$\ddot{x}_2 - 0.01\,(\dot{x}_1 - \dot{x}_2) - 2\,(x_1 - x_2) - (x_1 - x_2)^3$$
$$+ 0.01\,(\dot{x}_2 - \dot{x}_3) + 2\,(x_2 - x_3) + (x_2 - x_3)^3 = 0$$

$$2\ddot{x}_3 - 0.01\,(\dot{x}_2 - \dot{x}_3) - 2\,(x_2 - x_3) - (x_2 - x_3)^3$$
$$+ 0.01\,\dot{x}_3 + x_3 = 0$$

As in the two-degree-of-freedom problem, the fact that the damping coefficients are equal, the third spring is linear and the second and third masses are not forced, has no bearing on the difficulty of the numerical procedure. The solution to the problem by perturbation or iteration is complex and hence, none is available for comparison.

It is interesting at this point, to compare the CDC 6600 computer processing times for problems of one, two and three degrees of freedom. The processing times per second of the time axis for the three problems respectively were 0.63, 6.83 and 11.5 seconds. Thus it can be seen that as more degrees of freedom are considered, the computer time required increases rapidly.

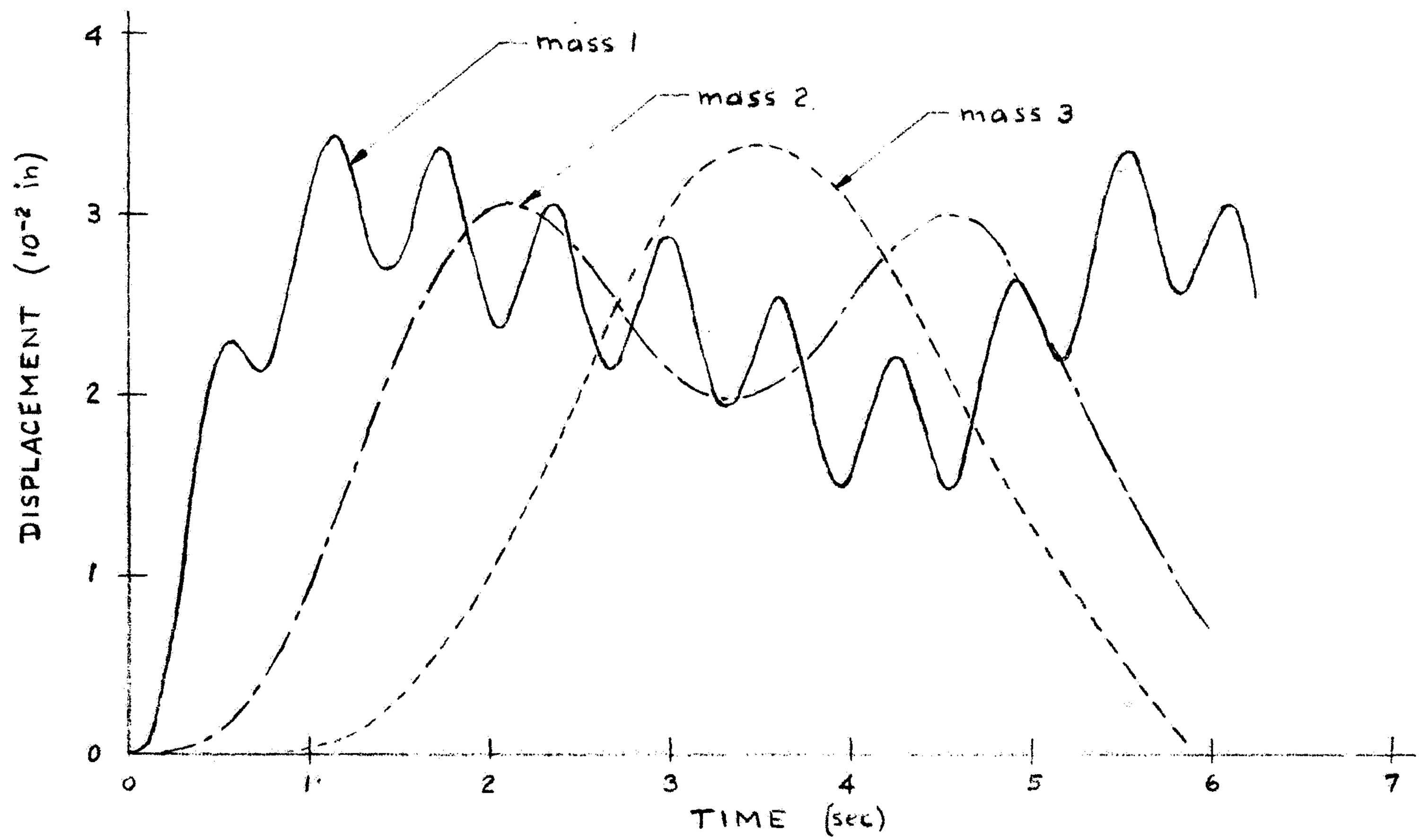

FIG. 10 - FORCED VIBRATION, 3 DEGREES OF FREEDOM

The dynamic response of a shallow arch or slightly curved beam to impulse loads is shown in Fig. 11. Three modes were considered in the solution so that the governing equations are Eq. (7) with N = 3. Certain basic dimensions of the arch were chosen arbitrarily. A span of 100 ft. and a rectangular cross-section, 1 ft. wide and 2 ft. deep were selected. The material properties were taken to be those of structural steel so that $E = 30 \times 10^6$ psi and W= 490 pcf. The rise of the arch was deliberately taken small so that an unstable condition would exist at a reasonable load level. Instability here refers to the phenominon of snap buckling which means that the arch reserves curvature and assumes the geometric configuration shown by the dashed line in Fig. 2. In terms of the notation used, the buckled state is equivalent to w less than zero. For the arch shown, the rise to span ratio was taken to be 0.02 so that the rise was 2 ft.

The distributed load p (x, t) was applied downward, that is in the negative sense, consistent with the sign convention adopted in Fig. 2. The spacial distribution is sinusoidal with respect to the x axis so that $q_2 = q_3 = 0$. In order that the undriven second and third modes be excited, small initial velocities, $\dot{a}_{2i}$ and $\dot{a}_{3i}$, are prescribed. All other initial conditions are zero. The time variation of the load is arbitrarily taken to be constant and is applied for a time equivalent to the fundamental, linear natural period of the simple beam. The load intensity for the condition shown in Fig. 11 is 10.8 k/ft.

The solid line in Fig. 11 shows the response of the structure with damping neglected. The maximum deflection is 24.5 in. or 1/2 in. larger than the

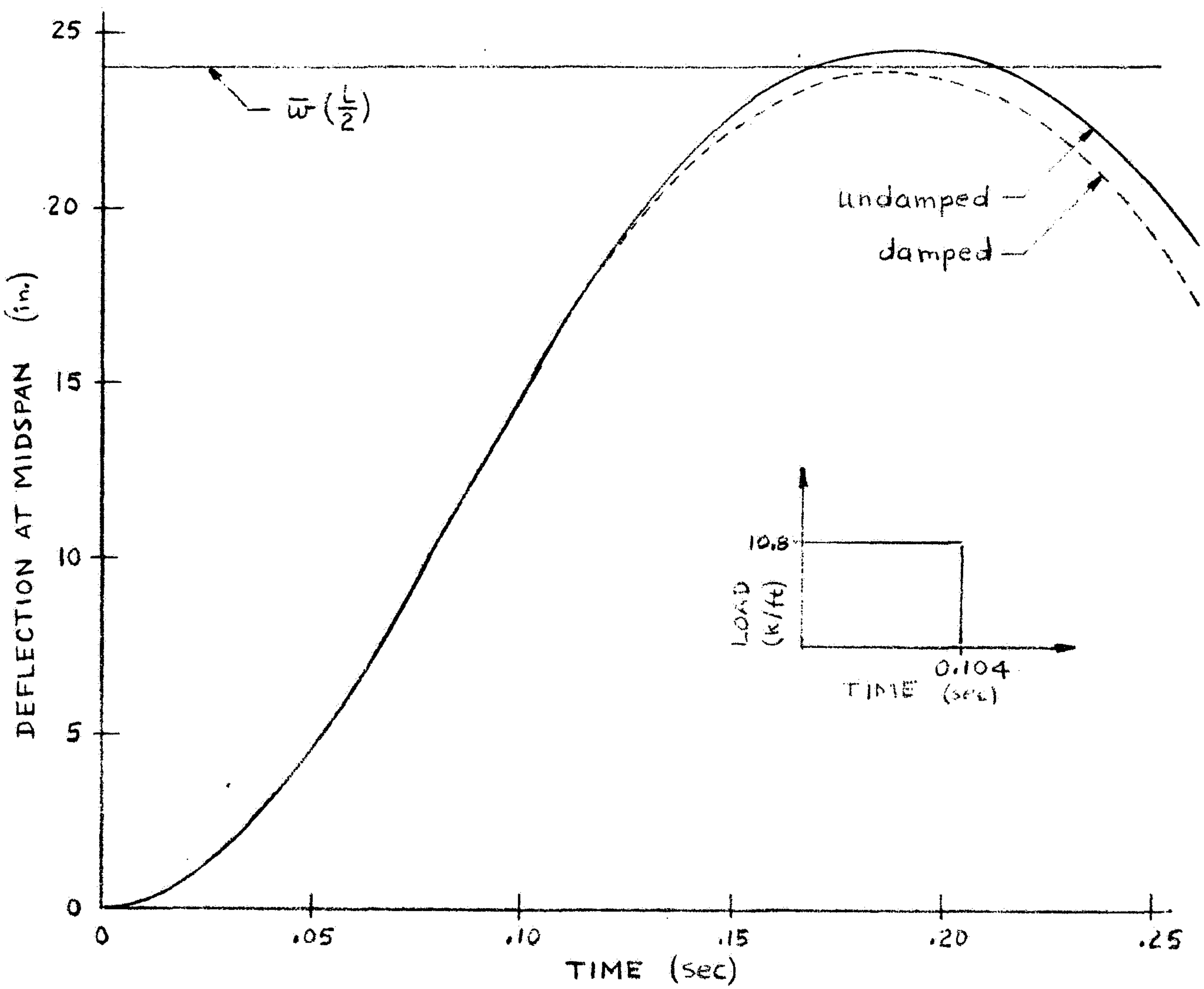

FIG. 11 — RESPONSE OF SHALLOW ARCH

initial rise. The dashed line shows the response of the same structure with small damping included. The damping coefficient was taken as $0.01 \left[2\rho A\omega_o\right]$. The maximum deflection of the damped system is 23.9 in. which is 0.1 in. less than the initial rise. The deflections of the damped system are slightly smaller than those of the undamped system until the load is removed at which point the accelerations change sign and the difference between the two deflections becomes significant.

It can be seen, as in the case of the discrete systems with softening springs, that instability can be detected by varying some critical parameter slightly and by comparing the results. The numerical procedure is seen to converge in all cases where an unstable condition exists.

VI. SUMMARY AND CONCLUSIONS

The dynamic response of nonlinear, elastic systems has been obtained by a step-by-step, numerical integration procedure. The effects of arbitrary initial conditions, linear damping and arbitrary loading are included. As opposed to most analytic methods, no assumed solution is required and the total response including transients and harmonics, is obtained. Both discrete and continuous systems have been considered.

The numerical results for a number of single-degree-of-freedom systems were compared to analytic, steady-state solutions obtained by the perturbation and iteration methods. It was shown that for the class of problems with small damping to which these methods are restricted, the steady-state solutions could be of little practical importance. The amplitude growth with time has been obtained numerically for several forced systems with damping and small, positive, cubic nonlinearities. For the sake of comparison with the analytic solutions, a harmonic forcing function was selected. The amplitudes were shown to approach the predicted, steady-state solutions as the transients in the system damped out. The numerical values of the coefficients in the governing equation were deliberately chosen in such a way that the analytic methods would converge for relatively large values of the damping coefficient. In systems with small or negligible damping, the transients may remain of significant importance for long periods of time.

The manner in which the amplitudes approached the steady-state values was found to depend on whether the frequency of the excitation was larger or smaller than the linear natural frequency of the system. When the driving frequency was larger than the linear natural frequency, the amplitudes were seen to approach the steady-state values asymptotically. For frequencies smaller than the natural frequency, the steady-state amplitudes were approached more rapidly and the total response exceeded the steady-state values before the transients had decayed sufficiently.

The free vibrations of a single-degree-of-freedom system with a softening or negative cubic nonlinearity were shown for several sets of initial conditions. Critical values of initial displacement and velocity at which the solutions cease to be periodic, were known from a phase plane analysis of the system. It was found that with initial conditions slightly less than and slightly greater than these critical values, the numerical procedure gave the periodic and nonperiodic response. The displacements in the case of the nonperiodic response, increased without bound. It was thus shown that instability, when present, could be detected by the numerical procedure and that the procedure converged for an unstable condition. For multi-degree-of-freedom systems where no topological analysis is possible, it would appear that the approach used here is the only way of treating such problems where the initial conditions determine the nature of the response.

A third class of single-degree-of-freedom problems with cubic nonlinearity but with no linear term was considered. The system was forced by a harmonic function and as in the previous problem, more than one solution was known to exist. A topological analysis of this system showed two stable periodic solutions and the range of initial conditions associated with each stable solution had been found. The amplitudes were known from analytic methods. Numerical results are shown for a damped and undamped system of this type. With zero initial conditions prescribed, the system with relatively large damping was seen to approach the proper periodic solution rapidly while the undamped system did not.

Numerical results for systems of two and three degrees of freedom have also been shown. No analytic solutions were obtained for comparison because of the complexity of the equations involved, but the results compare favorably with similar, single-degree-of-freedom systems.

As an example of how the numerical procedure can be applied to a continuous system, the dynamic response of a pin-ended, shallow sinusoidal arch to arbitrary impulse loading was obtained. The spacial variable was separated by the Galerkin Method and a system of nonlinear, ordinary differential equations in time was obtained. This system was then solved by the same numerical scheme used in analyzing the discrete systems. A three-mode approximation was used. The response of a particular arch to a constant impulse has been shown. A load intensity close to the critical value was found and the response of the structure with small damping was compared to the response with damping neglected. The undamped system was shown to buckle while the damped system remained stable. Numerical values for the maximum deflections were obtained.

As in the case of the discrete systems, the numerical integration procedure results in simultaneous, nonlinear algebraic equations. These systems of equations have been treated by the "method of continuity" which is described in Appendix C. The step-by-step integration inherent in the continuity scheme is, in turn, carried out by a fourth order Runge-Kutta procedure.

Although the numerical approach on the whole, does not take into account the nonuniqueness of the solutions to nonlinear systems, the method converged for all systems considered, to a physically meaningful and apparently correct solution. In the case of the arch as well as in other systems where more than one equilibrium position was known to exist, the method gave the appropriate solution for the set of numerical values chosen. In problems where an unstable response was obtained, the numerical procedure converged but smaller time increments were required as displacements and velocities increased.

Based on the problems which have been solved and the comparisons made, it appears that the numerical approach used, gives reliable results. In systems with little or no damping or where transient response is of interest, the results may be of greater value than those obtained by other methods. In practical, multi-degree-of-freedom systems, the numerical approach is the only means to a solution of any kind.

The nature of the numerical procedure and the simplicity of formulation raises the possibility of treating a number of other problems by the same basic approach. As has already been mentioned, the step-by-step integration procedure is ideally suited to the solution of variable coefficient problems. Time dependent coefficients in nonlinear systems could be handled in the same manner as was the arbitrary load in this presentation. Another interesting application of the method might result from its extension to two or three dimensions. By using different restoring forces in different directions, anisotropic systems could be simulated. There is also the feasibility of treating other types of nonlinearity. These would include material nonlinearity and nonlinear damping of which the self-oscillatory system is a special case. This problem has application in the theory of the flutter phenominon.

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APPENDIX A - Nomenclature

A	cross-sectional area
A_n, B_n	functions of solution at beginning of time interval
a_n	nondimensional Fourier coefficient
c	damping coefficient
E	modulus of elasticity
e	nondimensional arch rise
F	amplitude of excitation
f	function, subscript denoting end of time interval
I	moment of inertia
i	subscript denoting beginning of time interval
j	index
k_n	n^{th} spring constant
L	span
M	moment
m_n	n^{th} mass
N	number of degrees of freedom
n	index
P	load
p	distributed load
q	nondimensional load
R, R^*	system constants
r	radius of gyration
S, S^*	system constants
t	time

W	specific weight
w	displacement of arch
$\bar{w}$	unstressed displacement of arch
x	displacement, coordinate direction
z	coordinate direction
γ	nondimensional damping coefficient
ϵ	coefficient of nonlinear term
η	nondimensional displacement
$\bar{\eta}$	nondimensional unstressed displacement
λ	continuity parameter
ξ	nondimensional coordinate
ρ	volume mass density
τ	nondimensional time
ω	frequency of excitation
ω_o	fundamental linear natural frequency

APPENDIX B - Linear Acceleration Method

The equations of motion (1), describing the oscillatory system shown in Figure 1 are repeated here for convenience.

$$m_1 \ddot{x}_1 + c_1 (\dot{x}_1 - \dot{x}_2) + k_1 (x_1 - x_2) + \epsilon_1 (x_1 - x_2)^3 = P_1(t)$$

$$m_n \ddot{x}_n - c_{n-1} (\dot{x}_{n-1} - \dot{x}_n) - k_{n-1} (x_{n-1} - x_n) - \epsilon_{n-1} (x_{n-1} - x_n)^3$$

$$+ c_n (\dot{x}_n - \dot{x}_{n+1}) + k_n (x_n - x_{n+1}) + \epsilon_n (x_n - x_{n+1})^3 = P_n(t) \qquad \text{(B.1)}$$

$$n = 2, \ldots, N-1$$

$$m_N \ddot{x}_N - c_{N-1} (\dot{x}_{N-1} - \dot{x}_N) - k_{N-1} (x_{N-1} - x_N) - \epsilon_{N-1} (x_{N-1} - x_N)^3$$

$$+ c_N \dot{x}_N + k_N x_N + \epsilon_N x_N^3 = P_N(t)$$

The displacements and velocities are prescribed at the initial time t_i as

$$x_n(t_i) = x_{ni}$$

$$\dot{x}_n(t_i) = \dot{x}_{ni} \qquad n = 1, \ldots, N.$$

With the initial conditions prescribed, the initial accelerations $\ddot{x}_{ni}$ can be solved for directly from the Eqs. (B.1).

The solution at the initial time t_i can now be extended to $t = t_i + \Delta t$ by assuming that the accelerations vary linearly over the small time interval Δt. Expanding the displacements, velocities and accelerations in a Taylor Series about t_i yields

$$x_{nf} = x_{ni} + (\Delta t)\,\dot{x}_{ni} + \frac{(\Delta t)^2}{2}\,\ddot{x}_{ni} + \frac{(\Delta t)^3}{6}\,\dddot{x}_{ni} + \ldots$$

$$\dot{x}_{nf} = \dot{x}_{ni} + (\Delta t)\,\ddot{x}_{ni} + \frac{(\Delta t)^2}{2}\,\dddot{x}_{ni} + \ldots$$

$$\ddot{x}_{nf} = \ddot{x}_{ni} + (\Delta t)\,\dddot{x}_{ni} + \ldots$$

$$n = 1, \ldots, N$$

where $x_{nf} = x_n(t_f) = x_n(t_i + \Delta t)$. With derivatives up to order 3 retained, the expansion for $\ddot{x}_{nf}$ is linear. By solving the third expansion for $\dddot{x}_{ni}$ and substituting into the first two, the $\dddot{x}_{ni}$ can be eliminated and the result is

$$x_{nf} = x_{ni} + (\Delta t)\,\dot{x}_{ni} + \frac{(\Delta t)^2}{3}\left[\ddot{x}_{ni} + \frac{\ddot{x}_{nf}}{2}\right] \tag{B. 2a}$$

$$\dot{x}_{nf} = \dot{x}_{ni} + \frac{\Delta t}{2}\,(\ddot{x}_{ni} + \ddot{x}_{nf}) \tag{B. 2b}$$

$$n = 1, \ldots, N.$$

Eqs. (B. 2) can be rewritten as follows, solving for $\ddot{x}_{nf}$ in Eq. (B. 2a)

$$\ddot{x}_{nf} = \frac{6}{(\Delta t)^2}\left[x_{nf} - A_n\right] \tag{B. 3a}$$

$$\dot{x}_{nf} = \frac{\Delta t}{2}\,\ddot{x}_{nf} + \frac{B_n}{\Delta t} \tag{B. 3b}$$

where

$$A_n = x_{ni} + (\Delta t)\,\dot{x}_{ni} + \frac{(\Delta t)^2}{3}\,\ddot{x}_{ni}$$

$$B_n = \Delta t\,(\dot{x}_{ni} + \frac{\Delta t}{2}\,\ddot{x}_{ni})$$

$$n = 1, \ldots, N.$$

These equations correspond with Eqs. (2). The Eqs. (B. 3a), (B. 3b) and (B. 1) written at $t = t_f$ are three systems of equations for the three quantities x_{nf}, $\dot{x}_{nf}$ and $\ddot{x}_{nf}$ in terms of the known solution at $t = t_i$.

By substituting Eq. (B. 3a) into (B. 3b) and then into Eq. (B. 1) written at $t = t_f$, the accelerations $\ddot{x}_{nf}$ are eliminated and the following two systems of equations for the $\dot{x}_{nf}$ and x_{nf} result.

$$\dot{x}_{nf} = \frac{3}{\Delta t}\left[x_{nf} + (A_n - \frac{B_n}{3})\right] \qquad n = 1, \ldots, N \tag{B. 4}$$

$$\frac{6m_1}{(\Delta t)^2}\left[x_{1f} - A_1\right] = P_1(t_f) - c_1(\dot{x}_{1f} - \dot{x}_{2f}) - k_1(x_{1f} - x_{2f}) - \epsilon_1(x_{1f} - x_{2f})^3$$

$$\begin{aligned}\frac{6m_n}{(\Delta t)^2}\left[x_{nf} - A_n\right] = {} & P_n(t_f) + c_{n-1}(\dot{x}_{n-1,f} - \dot{x}_{nf}) + k_{n-1}(x_{n-1,f} - x_{nf}) \\ & + \epsilon_{n-1}(x_{n-1,f} - x_{nf})^3 - c_n(\dot{x}_{nf} - \dot{x}_{n+1,f}) \\ & - k_n(x_{nf} - x_{n+1,f}) - \epsilon_n(x_{nf} - x_{n+1,f})^3 \qquad n = 2, \ldots, N-1\end{aligned} \tag{B. 5}$$

$$\begin{aligned}\frac{6m_N}{(\Delta t)^2}\; x_N - A_N = {} & P_N(t_f) + c_{N-1}(\dot{x}_{N-1,f} - \dot{x}_{Nf}) + k_{N-1}(x_{N-1,f} - x_{Nf}) \\ & + \epsilon_{N-1}(x_{N-1,f} - x_{Nf})^3 - c_N\dot{x}_{Nf} \\ & - k_N x_{Nf} - \epsilon_N x_{Nf}^3\end{aligned}$$

The velocities $\dot{x}_{nf}$ can now be eliminated by substituting Eqs. (B. 4) into Eqs. (B. 5). Defining the system constants R_n and S_n as

$$R_n = \frac{3c_n}{\Delta t} \quad \text{and} \quad S_n = \frac{6m_n}{(\Delta t)^2},$$

the combining of Eqs. (B. 4) and (B. 5) yields the following system of N non-linear algebraic equations for the N displacements x_{nf}.

$$\left[S_1 + R_1 + k_1\right] x_{1f} - \left[R_1 + k_1\right] x_{2f} + \epsilon_1 (x_{1f} - x_{2f})^3$$

$$= S_1 A_1 + R_1 \left[(A_1 - A_2) - (\frac{B_1 - B_2}{3})\right] + P_1(t_f)$$

$$- \left[R_{n-1} + k_{n-1}\right] x_{n-1,f} + \left[S_n + R_{n-1} + R_n + k_{n-1} + k_n\right] x_{nf}$$

$$- \left[R_n + k_n\right] x_{n+1,f} - \epsilon_{n-1} (x_{n-1,f} - x_{nf})^3 + \epsilon_n (x_{nf} - x_{n+1,f})^3 \tag{B. 6}$$

$$= S_n A_n - R_{n-1} \left[(A_{n-1} - A_n) - (\frac{B_{n-1} - B_n}{3})\right]$$

$$+ R_n \left[(A_n - A_{n+1}) - (\frac{B_n - B_{n+1}}{3})\right] + P_n(t_f)$$

$$n = 2, \ldots, N-1$$

$$- \left[R_{N-1} + k_{N-1}\right] x_{N-1} + \left[S_N + R_{N-1} + R_N + k_{N-1} + k_N\right] x_{Nf}$$

$$- \epsilon_{N-1} (x_{N-1,f} - x_{Nf})^3 + \epsilon_N x_{Nf}^3$$

$$= S_N A_N - R_{N-1} \left[(A_{N-1} - A_N) - \left(\frac{B_{N-1} - B_N}{3}\right)\right] \tag{B.6}$$

$$+ R_N \left[A_N - \frac{B_N}{3}\right] + P_N(t_f)$$

Eqs. (B.6) correspond to Eqs. (3) of Section IV.

APPENDIX C - The Continuity Method

A number of various techniques are available for the solution of nonlinear algebraic equations. These techniques can become quite cumbersome however, in the treatment of large systems of such equations. It is pointed out by F. John (49) that in such cases, it is advantageous to differentiate the system of algebraic equations and to solve the resulting system of differential equations numerically.

This approach is also presented by Morrey in a volume edited by Beckenbach (50) where it is referred to as the method of continuity. Some of the following remarks are taken from this presentation.

The procedure consists of introducing a scalar variable λ, into the original algebraic system $f_n(x_1, \ldots, x_N)$ in such a way that the modified system

$$f_n(x_1, \ldots, x_N, \lambda) = 0 \tag{C.1}$$

is equal to the original system for $\lambda = 1$ and reduces to an easily solvable system for $\lambda = 0$. The essence of the method is now to choose the $x_1, \ldots, x_n$ as functions of λ so as to satisfy the modified equations for every λ in the interval $0 \leq \lambda \leq 1$.

With the x_j as functions of λ, the derivatives $x_j'(\lambda)$ would then satisfy the following equations obtained by differentiating Eqs. (C.1)

$$\sum_{j=1}^{N} \frac{\partial f_n}{\partial x_j}(x_1, \ldots, x_N, \lambda)\, x_j'(\lambda) + \frac{\partial f_n}{\partial \lambda}(x_1, \ldots, x_N, \lambda) = 0 \qquad \text{(C-2)}$$

$$n = 1, \ldots, N.$$

The Eqs. (C. 2) are N, linear ordinary differential equations in the N unknowns, $x_j(\lambda)$. The solution $x_j(1)$ to the original algebraic system is now obtained by integrating the $x_j(\lambda)$ from $\lambda = 0$ to $\lambda = 1$ starting with the known initial values, $x_j(0)$.

With regard to uniqueness, it is stated (50) that if the system can be solved for $\lambda = 0$ and if the partial derivatives in Eqs. (C. 2) are continuous, the solution exists and is unique for values of λ sufficiently near zero. The procedure can be continued uniquely until the solution encounters a singularity in the equations or goes off to infinity. If the solution can be continued to $\lambda = 1$, then the values of $x_j(1)$ are solutions to the original equations.